SOLUTIONS

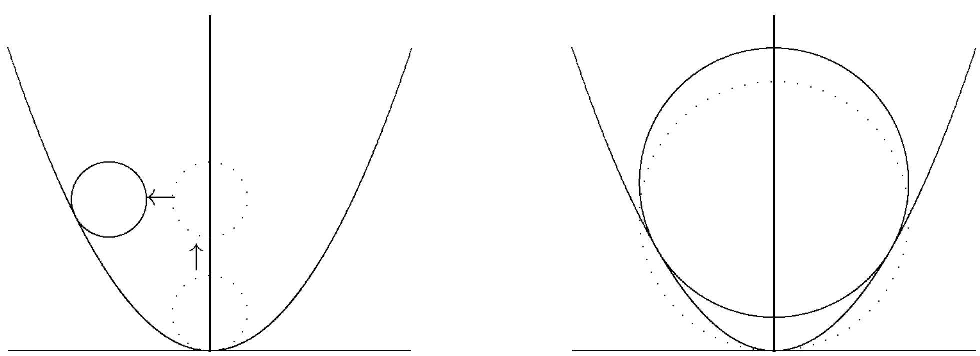
1. A circle rolls along the inside arc of the parabola $y = x^2$. What is the radius of the largest circle that will eventually reach the bottom of the parabola without getting stuck before getting there?

Answer. The largest circle that will roll to the bottom of the parabola is one with radius 1/2.

Solution. A circle through the origin and center on the positive y axis has the equation $x^2 + (y - r)^2 = r^2$, $r > 0$. It intersects the parabola $y = x^2$ at points x for which

$$x^2 + (x^2 - r)^2 = r^2$$
$$x^4 + (1 - 2r)x^2 = 0$$
$$x^2\big(x^2 + (1 - 2r)\big) = 0.$$

There will be three distinct intersection points if and only if $x^2 = 2r - 1$ has two distinct solutions, which happens if and only if $2r - 1 > 0$, or $r > 1/2$. So for $r \le 1/2$, the circle will be tangent to the parabola at the origin and will not intersect it at any other point. Such a circle will roll to the bottom of the parabola without getting stuck (that is, it will meet the parabola at one point, the point of tangency, along the path of the parabola; this can be seen by appropriately raising the circle through the origin and shifting it right or left to coincide with the circle on the parabolic path; see left diagram below).



For $r > 1/2$, the circle through the origin and center on the y-axis will intersect the parabola at three distinct points. So shifting this circle upward until it is tangent to the parabola, we see that this circle will not roll to the bottom of the parabola (the points of tangency are symmetric across the y-axis; see right diagram above).

We conclude that a circle will roll to the bottom of the parabola if and only if $r \le 1/2$, and the largest of these circles is $r = 1/2$.

2. Arrange numbers in an infinite array of three columns as defined recursively in the following manner. The first row is $[1,\ 2,\ 4]$, and for $n > 1$, row n is $[a,\ b,\ a+b+1]$, where a and b, with $a < b$, are the two smallest positive integers that have not yet appeared as entries in rows $1, 2, \ldots, n-1$. The first two rows of the array are

	Column 1	Column 2	Column 3
Row 1	1	2	4
Row 2	3	5	9

Note that after row 1 is given, 3 and 5 are the two smallest positive integers that have not yet been placed, so they appear in columns 1 and 2 of row 2. In which row and column will each of the following numbers appear?

a. 2007 b. 2008 c. 2009

Answer. 2007 is in row 803, column 2; 2008 is in row 804, column 1; 2009 is in row 402, column 3.

Solution. To get an idea, here are the first few rows of the array:

	Column 1	Column 2	Column 3
Row 1	1	2	4
Row 2	3	5	9
Row 3	6	7	14
Row 4	8	10	19
Row 5	11	12	24
Row 6	13	15	29
Row 7	16	17	34
Row 8	18	20	39
Row 9	21	22	44
Row 10	23	25	49
Row 11	26	27	54
Row 12	28	30	59

a. We see that

		Column 2
Row	$3 = 4 \cdot 0 + 3$	$7 = 10 \cdot 0 + 7$
Row	$7 = 4 \cdot 1 + 3$	$17 = 10 \cdot 1 + 7$
Row	$11 = 4 \cdot 2 + 3$	$27 = 10 \cdot 2 + 7$

and it's pretty clear this pattern will persist. Since $2007 = 10 \cdot 200 + 7$, we see that 2007 will occur in row $4 \cdot 200 + 3 = 803$ of the third column.

b. We see that

		Column 2
Row	$4 = 4 \cdot 0 + 4$	$8 = 10 \cdot 0 + 8$
Row	$8 = 4 \cdot 1 + 4$	$18 = 10 \cdot 1 + 8$
Row	$12 = 4 \cdot 2 + 4$	$28 = 10 \cdot 2 + 8$

and from this we conclude that 2008 will occur in the first column in row $4 \cdot 200 + 4 = 804$.

c. In this case

		Column 3
Row	$2 = 4 \cdot 0 + 2$	$9 = 10 \cdot 0 + 9$
Row	$4 = 4 \cdot 0 + 4$	$19 = 10 \cdot 1 + 9$
Row	$6 = 4 \cdot 1 + 2$	$29 = 10 \cdot 2 + 9$
Row	$8 = 4 \cdot 1 + 4$	$39 = 10 \cdot 3 + 9$
Row	$10 = 4 \cdot 2 + 2$	$49 = 10 \cdot 4 + 9$
Row	$12 = 4 \cdot 2 + 4$	$59 = 10 \cdot 5 + 9$

and because $2009 = 10 \cdot 200 + 9$, we conclude that 2009 occurs in column 3 in row $4 \cdot 100 + 2 = 402$.

Here's a more formal proof. The preceding table suggests the following pattern for all k:

	Column 1	Column 2	Column 3
Row $4k+1$	$10k+1$	$10k+2$	$10(2k)+4$
Row $4k+2$	$10k+3$	$10k+5$	$10(2k)+9$
Row $4k+3$	$10k+6$	$10k+7$	$10(2k+1)+4$
Row $4k+4$	$10k+8$	$10k+10$	$10(2k+1)+9$

To verify this pattern persists, we can induct on k. As we've seen, the pattern holds for $k = 0, 1, 2$, so suppose it's true for $k = 0, 1, 2, \ldots, K$. The induction hypothesis implies that first $4K+4$ rows contain all integers from 1 to $10(K+1)$, in addition to, in column 3, $10(K+1)+4$, $10(K+2)+4$, $\ldots$, $10(2K+1)+4$ and $10(K+1)+9$, $10(K+2)+9$, $\ldots$, $10(2K+1)+9$.

The two smallest positive integers that have not been placed yet are $10(K+1)+1$ and $10(K+1)+2$, so they must go into columns 1 and 2 of row $4(K+1)+1$, respectively, and their sum, $10\big(2(K+1)\big)+4$, goes into column 3. The next smallest number that hasn't yet appeared is $10(K+1)+3$ and so it will go in the first column of row $4(K+1)+2$. Now $10(K+1)+4$ has already been placed (in column 3), but $10(K+1)+5$ hasn't occurred, so it will go to column 2 in row $4(K+1)+2$. The next two numbers that haven't been taken are $10(K+1)+6$ and $10(K+1)+7$, so these will go in columns 1 and 2 of column $4(K+1)+3$, respectively. Then $10(K+1)+8$ will go into column 1 of row $4(K+1)+4$. Finally, since $10(K+1)+9$ has already been placed, but $10(K+1)+10$ hasn't, the latter will go into column 2 of row $4(K+1)+4$. In summary, the next four rows of the array are

	Column 1	Column 2	Column 3
Row $4(K+1)+1$	$10(K+1)+1$	$10(K+1)+2$	$10\big(2(K+1)\big)+4$
Row $4(K+1)+2$	$10(K+1)+3$	$10(K+1)+5$	$10\big(2(K+1)\big)+9$
Row $4(K+1)+3$	$10(K+1)+6$	$10(K+1)+7$	$10\big(2(K+1)+1\big)+4$
Row $4(K+1)+4$	$10(K+1)+8$	$10(K+1)+10$	$10\big(2(K+1)+1\big)+9$

Because this set of rows has the desired form for $k = K+1$, the proof is complete by induction.

3. Consider the set of all isosceles triangles whose base is on the non-negative x-axis (that is, $x \geq 0$) and whose (third) vertex is on the curve $y = x(4-x)^3$, $0 \leq x \leq 4$ $\big($for example, ΔABC, with $A = (1,0), B = (3,3), C = (5,0)\big)$. Among these triangles, which one has the largest area, and what is it?

Answer. The triangle of maximum area has base vertices at $(0,0)$ and $(16/5, 0)$, and its area is $\dfrac{2^{12}3^3}{5^5} = 35.38944$.

Solution. Suppose an isosceles triangle has its vertex at $\big(x, x(4-x)^3\big)$. The longest the base can be for this vertex is the one from $(0,0)$ to $(2x, 0)$ (on the x-axis), and the area of this triangle is $A(x) = x^2(4-x)^3$. We need to maximize this value over the interval $[0,4]$. Clearly, $A(0) = A(4) = 0$, and

$$A'(x) = 2x(4-x)^3 - 3x^2(4-x)^2 = x(4-x)^2\big(2(4-x) - 3x\big) = x(4-x)^2(8-5x).$$

The derivative is 0 for $x = 0, x = 4$, and $x = 8/5$. So the maximum area occurs at $x = 8/5$ (more formally, at that critical point the curve is concave down since $A''(8/5) = (4-x)^2(8-5x) - 2x(4-x)(8-5x) - 5x(4-x)^2\Big|_{x=8/5} = -8(4-8/5)^2 < 0$.)

Therefore, the maximum area is

$$A\left(\frac{8}{5}\right) = \left(\frac{8}{5}\right)^2\left(4 - \frac{8}{5}\right)^3 = \left(\frac{2^6}{5^2}\right)\left(\frac{12}{5}\right)^3 = \frac{2^6 \cdot 2^6 \cdot 3^3}{5^5} = \frac{2^{12}3^3}{5^5} = 35.38944.$$

4. Define a sequence of *positive* numbers by $a_1 = a_2 = 1$ and for $n \geq 2$,

$$(a_{n+1})^2 = 1 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}}\right) + \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{(a_{n-1})^2}.$$

Prove that a_n is a rational number for all n.

Solution. Applying the recursion formula,

$$(a_3)^2 = a_0 + 2\left(\frac{a_2}{a_1}\right) + \frac{1}{a_1^2} = 1 + 2 + 1 = 4, \qquad \text{so } a_3 = 2, \qquad \text{and}$$

$$(a_4)^2 = a_0 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2}\right) + \frac{1}{a_1^2} + \frac{1}{a_2^2} = 1 + 2(1+2) + (1+1) = 9, \qquad \text{so } a_4 = 3.$$

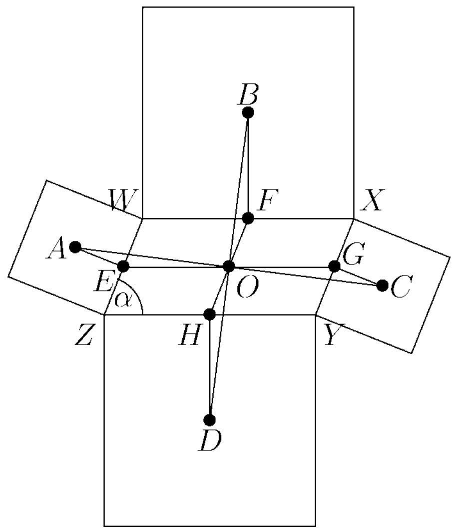
For $n \geq 4$,

$$\begin{aligned}
(a_{n+1})^2 &= 1 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_{n-1}}\right) + \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{(a_{n-1})^2} \\
&= 1 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_{n-1}}{a_{n-2}}\right) + 2\,\frac{a_n}{a_{n-1}} + \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{(a_{n-2})^2}\right) + \frac{1}{(a_{n-1})^2} \\
&= 1 + 2\left(\frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_{n-1}}{a_{n-2}}\right) + \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{(a_{n-2})^2}\right) + 2\,\frac{a_n}{a_{n-1}} + \frac{1}{(a_{n-1})^2} \\
&= a_n^2 + 2\,\frac{a_n}{a_{n-1}} + \frac{1}{(a_{n-1})^2} \\
&= \left(a_n + \frac{1}{a_{n-1}}\right)^2
\end{aligned}$$

so $a_{n+1} = a_n + \dfrac{1}{a_{n-1}}$. Now, assuming by induction that a_n and a_{n-1} are rational numbers, it follows that a_{n+1} is also a rational number, because the sum of two rational numbers is a rational number. Therefore, by induction, a_n is rational number for all n.

5. Squares are constructed outward on the respective sides of a parallelogram. Prove that the centers of these four squares are the vertices of a square.

Solution 1. (Synthetic geometry) In the figure, $WXYZ$ is a parallelogram, A, B, C, D are the centers of the squares, O is the center of the parallelogram, EG and FH are lines through O parallel to the sides of the parallelogram as shown, and α is the acute angle (the result is clearly true if $\alpha = 90°$) between adjacent sides of the parallelogram.



Then $|AE| = |OF| = |CG| = |OH| = \frac{1}{2}|WZ|$ and $|OE| = |BF| = |OG| = |DH| = \frac{1}{2}|YZ|$. Furthermore, $\angle AEO = \angle BEO = \angle CFO = \angle DHO = 90° + \alpha$, so triangles AOE, BOF, COG and DOH are congruent by SAS. But then $|AO| = |BO| = |CO| = |DO|$ because corresponding sides of congruent triangles are equal. Also, $\angle AOE = \angle COG$ and $\angle BOF = \angle COH$ so AOC and BOD are straight lines (EG and FH are straight lines). Finally,

$$\begin{aligned}\angle BOC &= \angle BOF + \angle FOG + \angle GOC \\ &= \angle BOF + \alpha + \angle EBO = (\angle BOF + \angle FBO) + \alpha \\ &= (180° - \angle BFO) + \alpha = (180° - (90° + \alpha) + \alpha = 90°.\end{aligned}$$

Putting this together, $ABCD$ is a quadrilateral whose diagonals intersect in a right angle and whose vertices are equidistant from the center; this implies that $ABCD$ is a square.

Solution 2. (Coordinate geometry) Referring to the figure in the first solution, let $F = (b, c)$ and $G = (a, 0)$. Then the coordinates of the vertices of the parallologram are

$$W = (-a + b, c), \qquad X = (a + b, c), \qquad Y = (a - b, -c), \qquad Z = (-a - c, -b),$$

and the coordinates of the centers of the squares are

$$A = (-a - c, b), \qquad B = (b, a + c), \qquad C = (a + c, -b), \qquad D = (-b, -a - c).$$

The latter shows that A and C are symmetric with respect to the origin O, and similarly for B and D, so the diagonals of $ABCD$ intersect at O. Also

$$|AO|^2 = (-a - c)^2 + b^2 = |BO|^2$$

so A, B, C, D are equidistant from the origin. Finally, the diagonals are perpendicular because their slopes are negative reciprocals:

$$\text{Slope } BD = \frac{a+c}{b} \qquad \text{and} \qquad \text{Slope } AC = \frac{-b}{a+c}.$$

Solution 3. (Vectors) Referring to the figure of the first solution, let $\mathbf{u}$ denote the vector from the origin O to the point F, and $\mathbf{v}$ the vector from O to G. Let $\mathbf{u}^\perp$ be the vector $\mathbf{u}$ rotated 90° clockwise about O, and $\mathbf{v}^\perp$ the vector $\mathbf{v}$ rotated 90° clockwise about O. Then the centers of the squares are given by the vectors

$$\mathbf{A} = -\mathbf{v} + \mathbf{u}^\perp, \qquad \mathbf{B} = \mathbf{u} + \mathbf{v}^\perp, \qquad \mathbf{C} = \mathbf{v} - \mathbf{u}^\perp, \qquad \mathbf{D} = -\mathbf{u} - \mathbf{v}^\perp$$

We see that $\mathbf{A}$ and $\mathbf{C}$ are negatives of each other so they have the same length. Also, $\mathbf{B}$ and $\mathbf{D}$ are negatives of each other and have the same length. Next, we calculate the lengths of $\mathbf{A}$ and $\mathbf{B}$; for this, let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$|\mathbf{A}|^2 = (-\mathbf{v}+\mathbf{u}^\perp)\cdot(-\mathbf{v}+\mathbf{u}^\perp) = |\mathbf{v}|^2 + |\mathbf{u}^\perp|^2 - 2|\mathbf{v}||\mathbf{u}^\perp|\cos(\theta+90°) = |\mathbf{v}|^2 + |\mathbf{u}|^2 + 2|\mathbf{v}||\mathbf{u}|\sin\theta$$

and

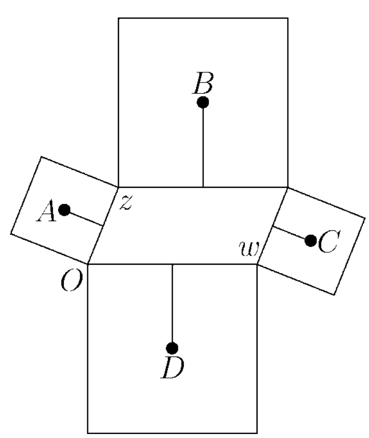
$$|\mathbf{B}|^2 = (\mathbf{u}+\mathbf{v}^\perp)\cdot(\mathbf{u}+\mathbf{v}^\perp) = |\mathbf{u}|^2 + |\mathbf{v}^\perp|^2 - 2|\mathbf{u}||\mathbf{v}^\perp|\cos(90° - \theta) = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}|\sin\theta$$

and therefore all four vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ have the same length. Finally,

$$\mathbf{B}\cdot\mathbf{C} = (\mathbf{u}+\mathbf{v}^\perp)\cdot(\mathbf{v}-\mathbf{u}^\perp) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u}^\perp + \mathbf{v}^\perp\cdot\mathbf{v} - \mathbf{v}^\perp\cdot\mathbf{u}^\perp = \mathbf{u}\cdot\mathbf{v} - \mathbf{v}^\perp\cdot\mathbf{u}^\perp = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} = 0,$$

so $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, and therefore the centers of the squares are the vertices of a square.

Solution 4. (Complex numbers) Orient the parallelogram in the complex plane so that one vertex is at the origin, let w and z be the coordinates of the vertices at the ends of the adjacent sides at that vertex as indicated, and let A, B, C, D be the centers of the squares on the sides of the parallelogram as shown.



Then the coordinates of the parallelogram are $0, w, z$ and $w + z$. The coordinates of the centers of the squares are

$$\begin{aligned} A &= \tfrac{1}{2}z + \tfrac{1}{2}ze^{\pi i/2} \\ B &= z + \tfrac{1}{2}w + \tfrac{1}{2}we^{\pi i/2} \\ C &= w + \tfrac{1}{2}z - \tfrac{1}{2}ze^{\pi i/2} \\ D &= \tfrac{1}{2}w - \tfrac{1}{2}we^{\pi i/2} \end{aligned}$$

Then a straight-forward calculation shows that

$$B - A = C - D = \tfrac{1}{2}(w + z) + \tfrac{1}{2}e^{\pi i/2}(w - z)$$

and

$$B - C = A - D = \tfrac{1}{2}(z - w) + \tfrac{1}{2}e^{\pi i/2}(z + w).$$

Also,

$$(B - A)e^{\pi i/2} = \left(\tfrac{1}{2}(w + z) + \tfrac{1}{2}e^{\pi i/2}(w - z)\right) e^{\pi i/2} = \tfrac{1}{2}(z - w) + \tfrac{1}{2}e^{\pi i/2}(z + w) = B - C,$$

which shows that the angle between AB and BC is a right angle and that the length of AB is equal to the length of BC (because the one side rotates into the other). In fact, because of the equalities we've already shown, the angles at each of the vertices A, B, C, D are right angles, and the sides have the same length, so it is a square.

6. For any two numbers a and b in the open interval $(-1, 1)$, let $\oplus$ be the binary operation defined by $a \oplus b = \dfrac{a + b}{1 + ab}$. For example, $1/2 \oplus 1/2 = 1/(1 + 1/4) = 4/5$. These numbers form a group under this operation. For this problem, all you need to know is that the operation is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all a, b, c in $(-1, 1)$. Here's the problem: For an arbitrary a in $(-1, 1)$, find a closed algebraic formula for $\underbrace{a \oplus a \oplus \cdots \oplus a}_{k}$.

Answer. $$\underbrace{a \oplus a \oplus \cdots \oplus a}_{k} = \frac{(1 + a)^k - (1 - a)^k}{(1 + a)^k + (1 - a)^k}$$

Solution. First, consider $k = 2, 3, 4, 5$.

$$a \oplus a = \frac{a + a}{1 + a^2} = \frac{2a}{1 + a^2},$$

$$a \oplus a \oplus a = \frac{a + \dfrac{2a}{1+a^2}}{1 + \dfrac{2a^2}{1+a^2}} = \frac{3a + a^3}{1 + 3a^2} \ ,$$

$$a \oplus a \oplus a \oplus a = \frac{a + \dfrac{3a + a^3}{1+3a^2}}{1 + \dfrac{3a^2 + a^4}{1+3a^2}} = \frac{4a + 4a^3}{1 + 6a^2 + a^4} \ ,$$

$$a \oplus a \oplus a \oplus a \oplus a = \frac{a + \dfrac{4a + 4a^3}{1+6a^2+a^4}}{1 + \dfrac{4a^2 + 4a^4}{1+6a^2+a^4}} = \frac{5a + 10a^3 + a^5}{1 + 10a^2 + 5a^4} \ .$$

These numbers look familiar! Recall that

$$(1+a)^5 = 1 + \underline{5a} + 10a^2 + \underline{10a^3} + 5a^4 + \underline{a^5} \tag{1}$$

and note that the sum of the underlined terms, the sum of the odd powers in the expansion, is the numerator of $a \oplus a \oplus a \oplus a \oplus a$, and the sum of the even powers is the denominator. So, we conjecture the formula given in the answer. We've seen that it holds for $k = 1, 2, 3, 4, 5$, so suppose the formula holds for k.

To prove this, return to consider $k = 5$. Note that

$$(1-a)^5 = 1 - 5a + 10a^2 - 10a^3 + 5a^4 - a^5 \tag{2}$$

and then by subtracting (2) from (1), we get twice the numerator for the case $k = 5$,

$$(1+a)^5 - (1-a)^5 = 2\Big(1 + 10a^2 + 5a^4\Big),$$

and adding (2) to (1), we get twice the denominator,

$$(1+a)^5 + (1-a)^5 = 2\Big(1 + 10a^2 + 5a^4\Big).$$

So this is the key. We have, by the inductive assumption,

$$\begin{aligned}
\underbrace{a \oplus a \oplus \cdots \oplus a}_{k+1} &= \frac{a + \dfrac{(1+a)^k - (1-a)^k}{(1+a)^k + (1-a)^k}}{1 + \dfrac{a(1+a)^k - a(1-a)^k}{(1+a)^k + (1-a)^k}} \\
&= \frac{a\Big((1+a)^k + (1-a)^k\Big) + \Big((1+a)^k - (1-a)^k\Big)}{\Big((1+a)^k + (1-a)^k\Big) + a\Big((1+a)^k - (1-a)^k\Big)} \\
&= \frac{(a+1)(1+a)^k + (a-1)(1-a)^k}{(1+a)(1+a)^k + (1-a)(1-a)^k} \\
&= \frac{(1+a)^{k+1} - (1-a)^{k+1}}{(1+a)^{k+1} + (1-a)^{k+1}}
\end{aligned}$$

and the induction is complete.

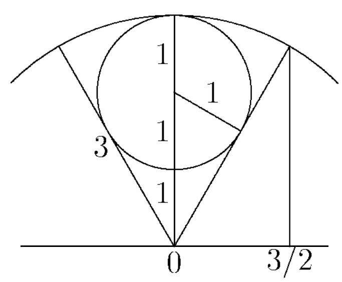
7. A solid ball of radius 1 is inside, and tangent to, a hollow sphere of radius 3. A light at the center of the sphere casts a shadow of the ball onto the sphere. Find the surface area of the shadowed region (on the encompassing sphere).

Answer. The surface area is $9\pi\left(2-\sqrt{3}\right) \doteq 7.576084931$.

Solution 1. (Two-variable calculus) This can be done with two-dimensional calculus by applying the formula for surface area of $z = f(x,y)$ over a region R:

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

In our case, $f(x,y) = \sqrt{9 - x^2 - y^2}$.



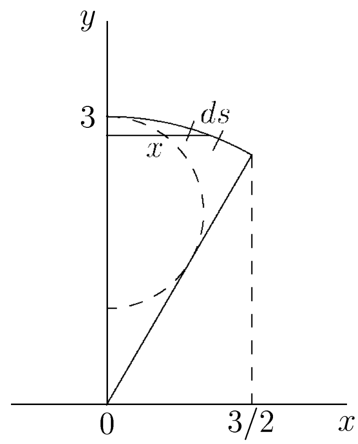
The angle subtended by the ball is 60°, so R is the region inside the circle centered at the origin whose radius is $3\sin 30° = 3/2$. Therefore, the surface area we want is give by the formula

$$\begin{aligned} A &= \iint_R \sqrt{1 + \left(\frac{-2x}{2\sqrt{9-x^2-y^2}}\right)^2 + \left(\frac{-2y}{2\sqrt{9-x^2-y^2}}\right)^2}\, dA \\ &= \iint_R \sqrt{1 + \left(\frac{x^2}{9-x^2-y^2}\right) + \left(\frac{y^2}{9-x^2-y^2}\right)}\, dA \\ &= \iint_R \frac{3}{\sqrt{9-x^2-y^2}}\, dA. \end{aligned}$$

Now, switching to polar coordinates, we have

$$\begin{aligned} A &= \int_0^{3/2}\int_0^{2\pi} \frac{3}{\sqrt{9-r^2}}\, r\, d\theta\, dr = 6\pi \int_0^{3/2} \frac{r\, dr}{\sqrt{9-r^2}} \\ &= -6\pi\sqrt{9-r^2}\,\Big|_0^{3/2} = -6\pi\left(\sqrt{9-9/4}\ - 3\right) \\ &= -3\pi\left(3\sqrt{3}-6\right) = 9\pi\left(2-\sqrt{3}\right) \doteq 7.576084931 \end{aligned}$$

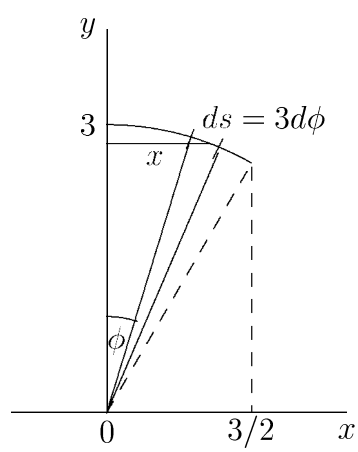
Solution 2. (One-variable calculus) The shadowed region is generated by revolving the arc $y = \sqrt{9 - x^2}$, $0 \le x \le 3/2$, about the y-axis.



Therefore, the area we want is

$$\begin{aligned} A &= \int_0^{3/2} (2\pi x) ds = 2\pi \int_0^{3/2} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_0^{3/2} x \sqrt{1 + \frac{x^2}{9 - x^2}}\, dx = 6\pi \int_0^{3/2} \frac{x\, dx}{\sqrt{9 - x^2}} \\ &= 9\pi \left(2 - \sqrt{3}\right) \qquad \text{(see the first solution).} \end{aligned}$$

Solution 3. (One-variable calculus) Using the same idea as in Solution 2, let ϕ be the angle shown in the figure, $0 \le \phi \le \pi/6$.



Then $ds = 3 \sin d\phi$ and $x = 3 \sin \phi$, so the integral is

$$\begin{aligned} A &= \int_0^{\pi/6} (2\pi(3\sin\phi))\, 3\, d\phi = 18\pi \int_0^{\pi/6} \sin\phi\, d\phi \\ &= -18\pi \cos\phi \Big|_0^{\pi/6} = -18\pi\,(\cos(\pi/6) - \cos 0) \\ &= -18\pi\left(\sqrt{3}/2 - 1\right) = 9\pi\left(2 - \sqrt{3}\right). \end{aligned}$$

8. Take a positive integer n_0, and add to it the number of odd digits and subtract the number of even digits. This gives a new number n_1. Now repeat this procedure starting with n_1 to get n_2, then continue with n_2 to get n_3, and so on. For example: $8 \to 7 \to 8 \to 7 \to \cdots$ is a cycle of length 2. Similarly, $11 \to 13 \to 15 \to 17 \to 19 \to 21 \to 21 \to \cdots$ ends as a cycle of length 1 (a fixed point), and $996 \to 997 \to 1000 \to 998 \to 999 \to 1002 \to 1000 \to 998 \to \cdots$ ends in a cycle of length 4. Will every starting number eventually end in a cycle? Prove or disprove.

Answer. Yes, every number must end in a cycle. It's easy to check that it's true for all single-digit numbers.

Idea. The result will follow if we can show the sequence is bounded. The jumps from one number to the next are small, so it can't get much beyond $200\ldots0$ for sufficient number of 0's, from whence the sequence will steadily decrease to a number below $200\ldots0$.

Solution. We'll argue by considering the number of digits in the starting number.

Consider 2-digit numbers. A 2-digit number that is not a fixed point under the transformation will change by 2, either up or down. The only way it could become a 3-digit number is for it to reach 99 (98 is a fixed point) and then go to 101, then to 102, and then back to 101, a cycle. Therefore 102 is an upper bound for the numbers in the chain. Because there are only finitely many numbers available, the chain will repeat, and that makes a cycle.

Now consider 3-digit numbers. Such a number might generate a 4-digit number (e.g., $999 \to 1002$). Note that any four digit number in the sequence will increase at the next step by at most 4, so if a number in the chain goes from less than 2000 to greater than or equal 2000, the latter number has to be one of $2000, 2001, 2002, 2003$. But the number at the next step will decrease, because there are more even digits than odd digits in each of these numbers. The numbers in the sequence will continue to decrease until a number is reached that is smaller than 2000, and then the terms may increase again, but our argument shows that none will exceed 2003.

Similarly, no element in a chain generated by a four-digit number can exceed 20004 (to go from less than 20000 to greater than or equal 20000 means it must take on one of the numbers $20000, 20001, 20002, 20003, 20004$, from where the sequence will decrease until it dips below 20000, and so forth).

In general, no k-digit number will produce a chain that gets beyond $B_k = 2 \cdot 10^{k+1} + k = 2\underbrace{00\ldots0}_{k}+k$. Again, this is because a $k+1$-digit number can increase by at most $k+1$, so to go from less that B_k to greater than or equal B_k, means that the latter number must be one of $B_k, B_k+1, \ldots B_k+k$. But these numbers each have more even digits than odd digits, so the sequence will decrease until it reaches a number less than B_k.

In summary, the numbers in the chain generated by n are bounded above by $2\underbrace{00\ldots0}_{k}+k$, where k is the number of digits in n. Because there are only finitely many numbers available, the sequence must eventually repeat, and this will make a cycle.

9. Take a permutation of the numbers 1 to 5 and consider the following procedure for sorting them into increasing order. Pick any number that's out of place, and wedge it to its "proper" position, shifting others over to make room for it. Repeat this procedure as long as there are numbers that are out of place. For example, take 3 5 1 2 4, and choose the underlined out-of-place number at each step.

$$\begin{array}{ccccc}
3 & 5 & 1 & 2 & \underline{4} \\
3 & 5 & 1 & 4 & \underline{2} \\
\underline{3} & 2 & 5 & 1 & 4 \\
2 & \underline{5} & 3 & 1 & 4 \\
2 & \underline{3} & 1 & 4 & 5 \\
\underline{2} & 1 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array}$$

In this case, it's taken 6 steps to sort the numbers into their proper order. It might have taken no more than 5 steps, had we considered the numbers in order $1, 2, 3, 4, 5$. On the other hand, it might have taken more steps.

a. Give an example of a permutation of $1, 2, 3, 4, 5$ that might require as many as 15 steps to sort using this procedure. Your example should give the sequence of steps by underlining the out-of-place numbers chosen for each step.

b. It turns out that this sorting procedure will always terminate in 15 steps or less. Prove that there are an *even* number of permutations that may take as many as 15 steps to sort with this algorithm.

Solution. a. Here's one permutation and choice of out-of-place numbers that takes 15 steps (one can be led to this by looking for the longest stretch of steps needed to sort $1, 2, 3$, or $1, 2, 3, 4$, by this algorithm).

$$
\begin{array}{ccccc}
5 & 1 & 2 & 3 & \underline{4} \\
5 & 1 & 2 & 4 & \underline{3} \\
5 & 1 & 3 & 2 & \underline{4} \\
5 & 1 & 3 & 4 & \underline{2} \\
5 & 2 & 1 & 3 & \underline{4} \\
5 & 2 & 1 & 4 & \underline{3} \\
5 & 2 & 3 & 1 & \underline{4} \\
5 & 2 & 3 & 4 & \underline{1}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\underline{2} & 3 & 4 & 5 & 1 \\
\underline{3} & 2 & 4 & 5 & 1 \\
\underline{2} & 4 & 3 & 5 & 1 \\
\underline{4} & 2 & 3 & 5 & 1 \\
\underline{2} & 3 & 5 & 4 & 1 \\
\underline{3} & 2 & 5 & 4 & 1 \\
\underline{2} & 5 & 3 & 4 & 1 \\
\underline{5} & 2 & 3 & 4 & 1
\end{array}
$$

$$
\begin{array}{ccccc}
1 & 5 & 2 & 3 & \underline{4} \\
1 & 5 & 2 & 4 & \underline{3} \\
1 & 5 & 3 & 2 & \underline{4} \\
1 & 5 & 3 & 4 & \underline{2} \\
1 & 2 & 5 & 3 & \underline{4} \\
1 & 2 & 5 & 4 & \underline{3} \\
1 & 2 & 3 & 5 & \underline{4} \\
1 & 2 & 3 & 4 & 5
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\underline{2} & 3 & 4 & 1 & 5 \\
\underline{3} & 2 & 4 & 1 & 5 \\
\underline{2} & 4 & 3 & 1 & 5 \\
\underline{4} & 2 & 3 & 1 & 5 \\
\underline{2} & 3 & 1 & 4 & 5 \\
\underline{3} & 2 & 1 & 4 & 5 \\
\underline{2} & 1 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{array}
$$

There are exactly 16 permutations (this is a task for a computer!) that may require 15 steps:

54132	54123	53412	51423	51234	45213	45132	45123
43521	43521	45231	34251	23451	35412	43512	34512

b. Let's say that 1 and 5 are "complements" of each other, that 2 and 4 are complements of each other, and 3 is its own complement. Also, let's say that the permutations $a\ b\ c\ d\ e$ and $e\ d\ c\ b\ a$ are "reverses" of each other. Denote the complement of a number a (between 1 and 5) by $\overline{a}$.

The key idea is to observe that if permutation $a\ b\ c\ d\ e$ requires 15 steps for a certain choice of out-of-place numbers, then the "reverse-complement" $\overline{e}\ \overline{d}\ \overline{c}\ \overline{b}\ \overline{a}$ is also a permutation that requires 15 steps for a certain choice of out-of-place numbers. An example will make this clear: Suppose we start with permutation 5 4 2 1 3. If, in the next step, we choose to place 4, we get the next permutation in the algorithm: 5 2 1 4 3. This corresponds to choosing 2 as the out-of-place number in the reverse-permutation: namely, from 3 5 4 2 1 (the reverse-complement) to 3 2 5 4 1. Now observe that the reverse-complement of this permutation is 5 4 2 1 3 (as we had obtained earlier).

This implies that the permutations that require 15 steps for a certain choice of out-of-place numbers come in pairs (mainly, a permutation and its reverse-complement), PROVIDED no permutation is equal to its reverse-complement. So suppose a permutation $a\ b\ c\ d\ e$ were equal to $\overline{e}\ \overline{d}\ \overline{c}\ \overline{b}\ \overline{a}$. Then $c = \overline{c}$, so $c = 3$. But observe that $a\ b\ 3\ d\ e$ could be reached, by means of the algorithm, from any of four different permutations: namely, $3\ a\ b\ d\ e$; $a\ 3\ b\ d\ e$; $a\ b\ d\ 3\ e$; $a\ b\ d\ e\ 3$. But this would mean that each of these permutations could require 16 steps to sort using appropriate out-of-place numbers along the way, and we're told that no permutation takes more than 15. This contradiction implies that no permutation requiring 15 steps is equal to its reverse-complement, and our proof is complete.

10. For each positive integer n, let $N(n) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \left\lceil \frac{n}{8} \right\rceil + \cdots + \left\lceil \frac{n}{2^k} \right\rceil$, where k is the unique integer such that $2^{k-1} \le n < 2^k$, and $\lceil x \rceil$ denotes the smallest number greater than or equal to x.

a. For which numbers n is $N(n) = n$?

b. Prove that your characterization in part a is correct.

Answer. (a.) There are at least two equivalent characterizations: (i) $N(n) = n$ if and only if the binary expansion of n consists of a string of 1's followed by a string of 0's (the string of 0's might be empty); (ii) $N(n) = n$ if and only if n can be written as $2^k - 2^\ell$ with $k > \ell$.

Solution. (b.) In binary notation, dividing by 2 just moves the decimal point one place to the left. So, if $d_k d_{k-1} \ldots d_2 d_1 d_0$ is the binary representation of n, then

$$N(n) = \lceil d_k d_{k-1} \ldots d_1 . d_0 \rceil + \lceil d_k d_{k-1} \ldots d_2 . d_1 d_0 \rceil + \cdots + \lceil d_k . d_{k-1} \ldots d_1 d_0 \rceil + \lceil . d_k d_{k-1} \ldots d_1 d_0 \rceil.$$

Now suppose that $n = \underbrace{11\ldots1}_{k}\underbrace{00\ldots0}_{\ell}$ in binary notation. Then

$$\begin{aligned}
N(n) &= \underbrace{11\ldots1}_{k}\underbrace{00\ldots0}_{\ell-1}.0 + \underbrace{11\ldots1}_{k}\underbrace{00\ldots0}_{\ell-2}.00 + \cdots + \underbrace{11\ldots1}.00\ldots0 + \\
&\quad \lceil \underbrace{11\ldots1}_{k-1}.1 \rceil + \lceil \underbrace{11\ldots1}_{k-2}.11 \rceil + \cdots + \lceil .11\ldots1 \rceil \\
&= \underbrace{11\ldots1}_{k}\left(2^{\ell-1} + 2^{\ell-2} + \cdots + 1\right) + \\
&\quad (\underbrace{11\ldots1}_{k-1}+1) + (\underbrace{11\ldots1}_{k-2}+1) + \cdots + (1+1) + 1 \\
&= \underbrace{11\ldots1}_{k}\left(2^\ell - 1\right) + \left(2^{k-1} + 2^{k-2} + \cdots + 1\right) \\
&= \left(2^k - 1\right)\left(2^\ell - 1\right) + \left(2^k - 1\right) \\
&= \left(2^k - 1\right)2^\ell = n.
\end{aligned}$$

To show that these are the only numbers that satisfy $N(n) = n$ we will prove that $N(n) = n +$ number of zeros between the first and last 1 in the binary representation of n (which proves that the described numbers satisfy the equation and they are the only ones that do so). For example, $N(11\underline{00}1\underline{0}1000) = 1100101000 + 3$, whereas $N(11110000) = 11110000$.

First, suppose that n has binary representation $n = d_k d_{k-1} \ldots d_2 d_1 1$, where $d_k = 1$. Then

$$\begin{array}{rllllllll} N(n) = & (d_k & d_{k-1} & d_{k-2} & \cdots & d_3 & d_2 & d_1) & +1+ \\ & & (d_k & d_{k-1} & \cdots & & d_3 & d_2) & +1+ \\ & & & (d_k & \cdots & & & d_3) & +1+ \\ & & & & \vdots & & & & \\ & & & & & & (d_k & d_{k-1}) & +1+ \\ & & & & & & & d_k & +1+ \\ & & & & & & & 0 & +1. \end{array}$$

and now adding along the diagonals,

$$\begin{aligned} N(n) &= d_k\left(1+2+2^2+\cdots+2^{k-1}\right) + d_{k-1}\left(1+2+\cdots+2^{k-2}\right) + \\ &\qquad d_{k-2}\left(1+2+\cdots+2^{k-3}\right) + \cdots + d_2\left(1+2\right) + d_1 \quad + (k+1) \\ &= d_k\left(2^k-1\right) + d_{k-1}\left(2^{k-1}-1\right) + d_{k-2}\left(2^{k-2}-1\right) + \cdots + d_1\left(2-1\right) + (k+1) \\ &= d_k\left(2^k-1\right) + d_{k-1}\left(2^{k-1}-1\right) + \cdots + d_1\left(2-1\right) + (1-1) + (k+1) \\ &= n - (d_k + d_{k-1} + d_{k-2} + \cdots + d_3 + d_2 + d_1 + 1) + (k+1) \end{aligned}$$

The second term is the sum of the coefficients of n which is just equal to the number of 1's in its binary representation. Since there are $k+1$ binary digits in the binary expression for n, $(k+1) - \left(d_k + d_{k-1} + \cdots + d_1 + 1\right)$ is just the number of 0's in the binary expression.

For the general case, suppose $n = m \cdot 2^s$, where m is odd. Then the binary form for m is $1xx\ldots x1$, for some binary digits $xx\ldots x$, so

$$\begin{aligned} N(n) &= m \cdot 2^{s-1} + m \cdot 2^{s-2} + \cdots + m + N(m) \\ &= m\left(2^{s-1} + 2^{s-2} + \cdots + 2 + 1\right) + N(m) \\ &= m\left(2^s - 1\right) + (m + \text{ the number of zeros in the binary expansion of } m) \\ &= n + \text{ the number of zeros between the first and last 1 in the binary form for } n. \end{aligned}$$

In summary, $N(n) = n$ if and only if the binary expansion of n consists of a string of 1's followed by a (possibly empty) string of 0's. Equivalently, $N(n) = n$ if and only if n is the difference of two powers of 2 $\left(\underbrace{11\ldots1}_{k}\underbrace{00\ldots0}_{s} = \underbrace{11\ldots1}_{k} \cdot 2^s = \left(2^k-1\right)2^s = 2^{k+s} - 2^s\right)$.