Twelfth Annual Iowa Collegiate Mathematics Competition

Luther College — March 4, 2006

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1. **Twelfth Anniversary Question!**

   Prove that $\displaystyle\int_0^1 x^{304}(1-x)^{2006}\,dx = \frac{304!\,2006!}{2311!}$.

   Sol. We prove, more generally for nonnegative integers m and n, that

$$\int_0^1 x^m(1-x)^n\,dx = \frac{m!\,n!}{(m+n+1)!}.$$

   Sol. #1. We apply induction on n. For $n=0$, it is easy to verify that

$$\int_0^1 x^m\,dx = \frac{1}{m+1} = \frac{m!\,0!}{(m+1)!}.$$

   Assume the result when $n-1$ is the exponent of $1-x$. We see that

$$\begin{aligned}\int_0^1 x^m(1-x)^n\,dx &= \int_0^1 x^m(1-x)^{n-1}\,dx - \int_0^1 x^{m+1}(1-x)^{n-1}\,dx\\ &= \frac{m!\,(n-1)!}{(m+n)!} - \frac{(m+1)!\,(n-1)!}{(m+n+1)!} = \frac{m!\,n!}{(m+n+1)!}.\end{aligned}$$

   Sol. #2. This time we apply induction on induction on m. For $m=0$, is easy to verify that

$$\int_0^1 (1-x)^n\,dx = \frac{1}{n+1} = \frac{0!\,n!}{(n+1)!}.$$

   Assume the result when $m-1$ is the exponent of x. Integrating by parts, we see that

$$\begin{aligned}\int_0^1 x^m(1-x)^n\,dx &= \frac{m}{n+1}\int_0^1 x^{m-1}(1-x)^{n+1}\,dx\\ &= \frac{m}{n+1}\cdot\frac{(m-1)!\,(n+1)!}{(m+n+1)!} = \frac{m!\,n!}{(m+n+1)!}.\end{aligned}$$

2. **No Fractions, Please**

   Let $p(x)$ be a polynomial with rational coefficients. Prove that there is an integer k such that $p(x+k)-p(x)$ is a polynomial with integer coefficients.

   Sol. #1. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and let k be a common multiple of the denominators of $a_0, a_1, \ldots, a_n$.

$$\begin{aligned} p(x+k)-p(x) &= \sum_{j=1}^{n} a_j \left((x+k)^j - x^j\right) = \sum_{j=1}^{n} a_j \sum_{i=1}^{j} \binom{j}{i} k^i x^{j-i} \\ &= \sum_{j=1}^{n} (ka_j) \sum_{i=1}^{j} \binom{j}{i} k^{i-1} x^{j-i}. \end{aligned}$$

Because ka_j is an integer, every term in this sum is an integer.

Sol. #2. Again, let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ and let k be a common multiple of the denominators of $a_0, a_1, \ldots, a_n$.

Using the difference of powers factorization

$$\begin{aligned} &p(x+k)-p(x) \\ &= a_1((x+k-x) + a_2\left((x+k)^2 - x^2\right) + \cdots + a_n\left((x+k)^n - x^n\right) \\ &= ka_1 + ka_2\left((x+k)+x\right) + \cdots \\ &\ +ka_n\left((x+k)^{n-1} + (x+k)^{n-2}x + \cdots + (x+k)x^{n-2} + x^{n-1}\right). \end{aligned}$$

Because ka_j is an integer for all j, every term in the latter expansion has integer coefficients. Therefore $p(x+k)-p(x)$ has integer coefficients.

3. **Maximal Distance to $\mathbf{y = x^p}$**

For what positive real numbers p is the maximal distance from the point $(1,0)$ to the curve $y = x^p$, $0 \le x \le 1$, equal to 1?

Sol. The maximal distance equals 1 for $p \ge 1/2$.

Sol. #1. The maximal distance exceeds 1 if and only if the curve $y = x^p$ passes above the quarter circle $(x-1)^2 + y^2 = 1$, $0 \le x \le 1$.This is equivalent to

$$x^p > \sqrt{1-(x-1)^2}$$

or

$$x^{2p} + x^2 - 2x > 0$$

for some x in $(0,1)$. For $p \ge 1/2$, both x^{2p} and x^2 are at most x for all x in $[0,1]$, so $x^{2p} + x^2 - 2x \le 0$. On the other hand,

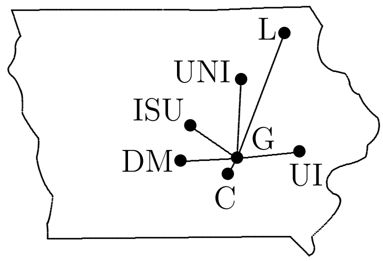
$$x^{2p} + x^2 - 2x = x^{2p}\left(1 + x^{2-2p} - 2x^{1-2p}\right).$$

For $p < 1/2$ and x sufficiently small, it follows that $1 + x^{2-2p} - 2x^{1-2p} \approx 1 > 0$, so that the maximal distance exceeds 1.

Sol. #2. We look at the square of the distance from (x, x^p) to $(1,0)$: $f(x) = (1-x)^2 + x^{2p}$. The derivative of f is $f'(x) = 2\left(px^{2p-1} + x - 1\right)$. If $p < 1/2$, then $2p-1 < 0$ and $f'(x) > 0$ for sufficiently small positive values of x. Therefore, f increases for sufficiently small positive values of x and has a maximum greater than $f(0) = 1$. On the other hand, for $p \ge 1/2$, $f(x) \le (1-x) + x = 1$.

4. **Corny Random Exchange Problem**

Seven math majors representing each of the hosts and winners of the Iowa Collegiate Mathematics Contest decide to participate in an exchange program. The seven students represent Iowa State University, Central College, Luther College, the University of Northern Iowa, Grinnell College, the University of Iowa, and the Des Moines Area Community Colleges (see map below).



First, the student at Grinnell randomly exchanges places with one of the other six students, with each equally likely to be chosen. The student now at Grinnell (who at this point is not originally from Grinnell) then exchanges places with one of the other six students, each again equally likely to be chosen. In all, there are six random exchanges of whichever student is currently at Grinnell and one of the other six students. What is the probability that each student ends up at his or her original location? Express your answer as a fraction in the form $m/6^r$.

Sol. The probability is $151/6^5$.

We look at three cases, depending on what happens in the first three exchanges. First of all, students at three different schools could exchange places with the students at Grinnell. In order for these students to return to their original schools, the same three exchanges must take place in reverse order. The probability of all this is

$$\frac{5}{6} \cdot \frac{4}{6} \cdot \left(\frac{1}{6}\right)^3 = \frac{20}{6^5}.$$

The second type of sequence of exchanges is when the first and third exchanges are between the same two schools and the second exchange involves a different school (and Grinnell as always). The net effect is that the students at the two schools other than Grinnell exchange places. The third type of exchange is when the school, in addition to Grinnell, involved in the second exchange is also involved in the first or third exchange, or both. The effect of this type is the same as that of the Grinnell student exchanging places with one other student. Thus, to undo three exchanges falling into the second type requires three exchanges of the second type involving the two students who have exchanged places. The probability of such a sequence of exchanges is

$$\frac{5}{6} \cdot \frac{1}{6} \cdot \frac{2}{6} \cdot \left(\frac{1}{6}\right)^2 = \frac{10}{6^5}.$$

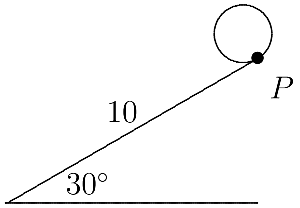
To undo three exchanges of the third type requires an exchange of this type where the same student, along with the Grinnell student, ends up moved. The probability of such a sequence of exchanges is

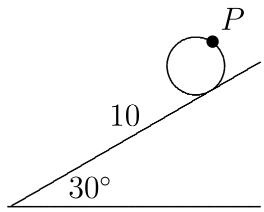
$$6 \cdot \left(\frac{11}{6^3}\right)^2 = \frac{121}{6^5}.$$

Summing these three probabilities yields the total probability of $151/6^5$.

5. **Rolling Down a Ramp**

A circle of radius 1 rolls down a straight ramp of length 10 which has angle of elevation 30°. If the point P on the circle is initially tangent to the ramp, find the maximum height off the ground of the point P.





Sol.

Clearly the maximum occurs in the first full revolution of the circle. (We note that the bottom point on the circle has y-coordinate greater than

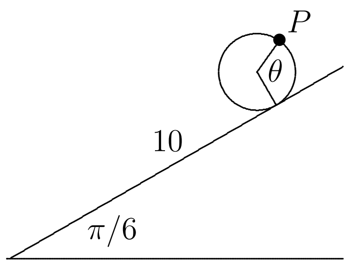
$$(10 - 2\pi)\sin \pi/6 - 1 = 4 - \pi > 0$$

for the first revolution, so the circle rotates more than once before reaching the bottom of the ramp.) We work in radian measure to make differentiation easier. We break the motion of the point down into the linear motion of the center of the circle and the rotation of P about this center. The center of the circle is initially

$$(10\cos \pi/6, 10\sin \pi/6) + (-\sin \pi/6, \cos \pi/6) = (10\cos \pi/6 - \sin \pi/6, 10\sin \pi/6 + \cos \pi/6).$$

When the point has rotated an angle θ, the point of tangency and the center of the circle will have moved a distance θ along the ramp, so that the center is at

$$((10 - \theta)\cos \pi/6 - \sin \pi/6, (10 - \theta)\sin \pi/6 + \cos \pi/6).$$



Relative to the center of the circle, P has polar coordinates $(1, \theta - \pi/3)$. Thus P will be at

$$\begin{aligned}
&((10 - \theta)\cos \pi/6 - \sin \pi/6, (10 - \theta)\sin \pi/6 + \cos \pi/6) + (\cos(\theta - \pi/3), \sin(\theta - \pi/3)) \\
&= ((10 - \theta)\cos \pi/6 - \sin \pi/6 + \cos(\theta - \pi/3), (10 - \theta)\sin \pi/6 + \cos \pi/6) + \sin(\theta - \pi/3)) \\
&= ((10 - \theta)\cos \pi/6 - \sin \pi/6 + \sin(\theta + \pi/6), (10 - \theta)\sin \pi/6 + \cos \pi/6 - \cos(\theta + \pi/6)).
\end{aligned}$$

We want to maximize

$$y(\theta) = (10 - \theta)\sin \pi/6 + \cos \pi/6 - \cos(\theta + \pi/6)$$

on $[0, 2\pi]$. We have

$$y' = -\sin \pi/6 + \sin(\theta + \pi/6).$$

The critical points are when $\theta = 0$ and when $\theta + \pi/6 = 5\pi/6$, i.e. $\theta = 2\pi/3$. Furthermore y increases on $[0, 2\pi/3]$ and decreases on $[2\pi/3, 2\pi]$. Therefore, the maximum value of y is

$$y(2\pi/3) = (10 - 2\pi/3)\sin \pi/6 + 2\cos \pi/6 = 5 - \pi/3 + \sqrt{3}.$$

6. **Factorial Divisibility**

Prove that $(n^2)!$ is divisible by $(n!)^{n+1}$ for all positive integers n.

Sol. #1. The number of ways of forming n distinct n-member committees out of n^2 people is the multinomial coefficient

$$\frac{(n^2)!}{(n!)^n}.$$

To count the number of ways of forming n indistinct n-member committees out of n^2 people, we must divide this multinomial coefficient by $n!$, obtaining

$$\frac{(n^2)!}{(n!)^{n+1}},$$

which must be an integer.

Sol. #2. We express $(n^2)!$ as the product of the n terms

$$[(k-1)n+1]\cdot[(k-1)n+2]\cdots[(k-1)n+(n-1)]\cdot[kn]$$

and $(n!)^{n+1}$ as the product of the n terms $k\cdot n!$, with k ranging from 1 to n in both cases. We see that it suffices to prove that

$$[(k-1)n+1]\cdot[(k-1)n+2]\cdots[(k-1)n+(n-1)]\cdot[kn]$$

is divisible by $k\cdot n!$. To see this, observe that

$$\begin{aligned}&\frac{[(k-1)n+1]\cdot[(k-1)n+2]\cdots[(k-1)n+(n-1)]\cdot[kn]}{k\cdot n!}\\ &=\frac{[(k-1)n+1]\cdot[(k-1)n+2]\cdots[(k-1)n+(n-1)]}{(n-1)!}\\ &=\binom{(k-1)n+(n-1)}{n-1}.\end{aligned}$$

Sol. #3 The number of $1,2,\ldots,m$ that are divisible by k is $\lfloor m/k\rfloor$, where $\lfloor\cdot\rfloor$ denotes the greatest integer function. Let p be a prime number and suppose j is the integer for which $p^j\le n<p^{j+1}$. It follows that the power of p dividing $(n^2)!$ is

$$\begin{aligned}&\left\lfloor\frac{n^2}{p}\right\rfloor+\left\lfloor\frac{n^2}{p^2}\right\rfloor+\cdots+\left\lfloor\frac{n^2}{p^j}\right\rfloor+\left\lfloor\frac{n^2}{p^{j+1}}\right\rfloor+\cdots+\left\lfloor\frac{n^2}{p^{2j}}\right\rfloor+\left\lfloor\frac{n^2}{p^{2j+1}}\right\rfloor\\ &\ge n\left\lfloor\frac{n}{p}\right\rfloor+n\left\lfloor\frac{n}{p^2}\right\rfloor+\cdots+n\left\lfloor\frac{n}{p^j}\right\rfloor+\left\lfloor\frac{n}{p}\right\rfloor+\cdots+\left\lfloor\frac{n}{p^j}\right\rfloor+0\\ &=(n+1)\left(\left\lfloor\frac{n}{p}\right\rfloor+\cdots+\left\lfloor\frac{n}{p^j}\right\rfloor\right).\end{aligned}$$

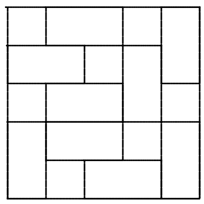
This last expression is just the power of p dividing $(n!)^{n+1}$, proving the claim.

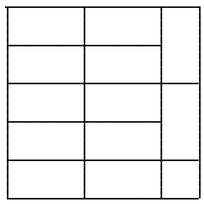
7. **Domino Tiling Game**

Consider the following game played on a 5×5 checkerboard. In turn, each player places a 1×2 domino so that it exactly covers two squares of the checkerboard. The last player able to place a domino wins. What are the shortest and longest possible games?

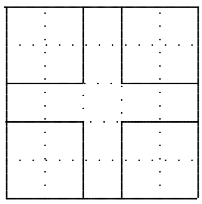
Sol. If we call the placement of a domino a "turn," then the shortest game lasts 9 turns and the longest game lasts 12 turns.

The following configurations show games of 9 and 12 turns, respectively.

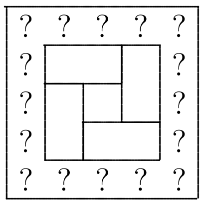




Clearly no game can last 13 turns, so we need only show that no game can end within 8 turns. Suppose some game ends after 8 or fewer turns. At least two dominoes must intersect each of the four the 2×2 corner blocks.



Because no single domino can intersect two of these 2×2 corner blocks, we have 8 dominoes accounted for and the center square of the 5×5 checkerboard is not covered. Therefore, the four squares adjacent to this center square must be covered, and, up to reflection, the board looks like the following.



There are four dominoes in the 16 border squares. Thus, the border contains 8 empty squares divided into at most 4 parts. Thus, there is room for another domino and the game is not over. This contradiction implies no game ends in 8 or fewer turns.

8. **Integer Points on a Folium of Descartes**

Find *all* pairs of integers (x, y) that satisfy $x^3 + y^3 = 6xy$.

Sol. #1. The only solutions are $(0, 0)$ and $(3, 3)$.

First observe that if $x = y$ then $x^3 = 3x^2$, implying $x = 0$ or 3, yielding the solutions $(0, 0)$ and $(3, 3)$. Now observe that if x or y is 0, so is the other. Finally, observe that x and y cannot both be negative. Thus, by symmetry it suffices to show that the two cases $x > y > 0$ and $x > 0 > y$ have no solutions.

If $x > y > 0$, by the arithmetic mean-geometric mean inequality

$$6xy = x^3 + y^3 > 2x^{3/2}y^{3/2},$$

hence $9 > xy$. If some prime number divides either x or y, then it divides the other. Thus, we need only do the arithmetic to check that $(4, 2)$ is not a solution. Therefore, there are no solutions satisfying $x > y > 0$.

When $x > 0 > y$, write $y = -z$. Then $x^3 + 6xz = z^3$, so that $x < z$. Substituting $z = x + k$ yields $(3k - 6)x^2 + (3k^2 - 6k)x + k^3 = 0$. However, this expression is clearly negative for $k = 1$ and clearly positive for $k > 1$. Hence, there are no solutions to $x^3 + y^3 = 6xy$ with $x > 0 > y$.

Sol. #2. Let p be a prime number and suppose x is divisible by p^j and y is divisible by p^k. From $x^3 = 6xy - y^3$, it follows that

$$3j \geq \min\{j + k, 3k\}.$$

Consequently $2j \geq k$ and similarly $2k \geq j$. From this or by direct check, it follows that if x or y is 0, so is the other. For $xy \neq 0$, it follows that x^2/y and y^2/x are (nonzero) integers. Because $x^2/y + y^2/x = 6$, their greatest common divisor d is a factor of 6. Now $x^2/(dy)$ and $y^2/(dx)$ are relatively prime integers whose quotient is x^3/y^3, i.e. each is a perfect cube. Writing $x^2/(dy) = a^3$ and $y^2/(dx) = b^3$, we have $a^3 + b^3 = 6/d$. No cube of absolute value greater than 1 is within 6 of another cube. We conclude that $a = b = 1$ and $d = 3$, which leads to the only nonzero solution, $x = y = 3$.

9. **A Functional Equation**

Find all differentiable functions f for which

$$f(x^3) - f(y^3) = (x^2 + xy + y^2)(f(x) - f(y))$$

for all real numbers x and y.

Sol. A function f satisfies the condition if and only if it is linear, i.e. $f(x) = a + bx$ for some constants a and b. It is easy to check that such a function satisfies the condition.

Conversely, for $x \neq 0$ and $y = 0$, dividing the equation by x^3 yields

$$\frac{f(x^3) - f(0)}{x^3} = \frac{f(x) - f(0)}{x}.$$

Replacing x with $x^{1/3}$, this implies

$$\frac{f(x) - f(0)}{x} = \frac{f(x^{1/3}) - f(0)}{x^{1/3}} = \frac{f(x^{1/9}) - f(0)}{x^{1/9}} = \cdots = \frac{f(x^{1/3^n}) - f(0)}{x^{1/3^n}}$$

for all positive integers n. We have

$$\lim_{n\to\infty} x^{1/3^n} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

It follows by continuity that

$$\frac{f(x) - f(0)}{x} = \begin{cases} f(1) - f(0) & \text{if } x > 0 \\ f(0) - f(-1) & \text{if } x < 0 \end{cases}$$

and that

$$f(x) = \begin{cases} f(0) + (f(1) - f(0))x & \text{if } x > 0 \\ f(0) & \text{if } x = 0 \\ f(0) + (f(0) - f(-1))x & \text{if } x < 0. \end{cases}$$

The differentiability of f at 0 now implies

$$f(1) - f(0) = f(0) - f(-1) = f'(0),$$

so that

$$f(x) = f(0) + f'(0)x$$

for all real x.

10. **Unsolvable**

Let $y = y(t)$ be a solution to the differential equation $y' + 2ty = t^2$. Evaluate

$$\lim_{t\to\infty} \frac{y}{t}.$$

Sol. The limit is 1/2.

The integrating factor for this differential equation is any function of the form $e^{\int 2t\,dt}$, say e^{t^2}. Multiplying the differential equation by e^{t^2} yields

$$\frac{d}{dt}\left(e^{t^2} y\right) = t^2 e^{t^2}.$$

We choose

$$\int_0^t \tau^2 e^{\tau^2}\, d\tau$$

for the antiderivative of $t^2 e^{t^2}$, so that

$$e^{t^2} y = \int_0^t \tau^2 e^{\tau^2}\, d\tau + C.$$

In fact, substituting $t = 0$, we see that $C = y(0)$. Therefore,

$$\frac{y}{t} = \frac{\int_0^t \tau^2 e^{\tau^2}\, d\tau + y(0)}{te^{t^2}}.$$

We use L'Hôpital's rule and the fundamental theorem of calculus to obtain

$$\lim_{t\to\infty} \frac{y}{t} = \lim_{t\to\infty} \frac{t^2 e^{t^2}}{(2t^2+1)e^{t^2}} = \lim_{t\to\infty} \frac{t^2}{2t^2+1} = \frac{1}{2}.$$