29th Annual Iowa Collegiate Mathematics Competition

Saturday, April 1, 2023

Solutions

Problems written by Damiano Fulghesu, Minnesota State University Moorhead

1. **Penny Game.** Aiden and Beatrix played a game of matching pennies. On each toss, Aiden won a penny from Beatrix if their coins matched, and Beatrix won one from Aiden if they failed to match. When they stopped, their coins had matched 13 times and Beatrix ended up with 8 more pennies than she started with. How many times did they toss?

   Solution.

   Let x be the number of tosses. Aiden won 13 of them, so Beatrix won the other $x-13$. Then Beatrix's net gain was $(x-13)-13 = x-26$ pennies. Therefore, we have:

   $$\begin{aligned} x - 26 &= 8 \\ x &= 34 \end{aligned}$$

   In conclusion, Aiden and Beatrix toss 34 times.

2. **Straight Motion.** A particle is moving along a straight line so that its velocity at time t, in seconds, is $3t^2$ meters per second. At what time t during the interval from $t = 0$ seconds and $t = 9$ seconds is its velocity the same as its average velocity over the entire interval?

   Solution.

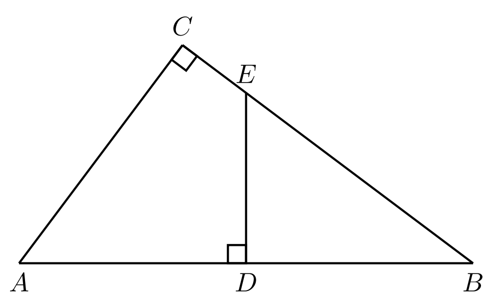
   The velocity is equal to the average velocity at $t = 3\sqrt{3}$ seconds.

   The total distance d traveled is given by the following integral:

   $$d = \int_0^9 3t^2 \, dt = t^3\Big|_0^9 = 9^3.$$

   Therefore, the average velocity is $\dfrac{9^3}{9} = 9^2 = 81$ meters per second. The velocity of the particle is 81 meters per second when $3t^2 = 81$, that is to say when $t = \sqrt{27} = 3\sqrt{3}$.

3. **Quadrilateral in a Triangle.** In the figure below, $AB = 20$, $AC = 12$, $AD = DB$, angles ACB and ADE are right angles. Determine the area of the quadrilateral $ADEC$.



   Solution.

   The area is 58.5. Let h be the altitude ED. From the Pythagorean Theorem, we have:

   $$BC = \sqrt{AB^2 - AC^2} = \sqrt{400 - 144} = \sqrt{256} = 16.$$

   Moreover, since $AD = BD$, we have $BD = 10$.

Now, the triangles ACB and EDB are similar since they are both right triangles and have an angle in common. Therefore, we have the identity:

$$\begin{aligned} \frac{ED}{BD} &= \frac{AC}{BC} \\ \frac{h}{10} &= \frac{12}{16} \\ h &= \frac{12}{16}\cdot 10 = \frac{15}{2} \end{aligned}$$

The area of triangle ACB is

$$\frac{1}{2}\cdot 12\cdot 16 = 96,$$

and the area of triangle EDB is

$$\frac{1}{2}\cdot\frac{15}{2}\cdot 10 = \frac{75}{2} = 37.5.$$

Then the area of the quadrilateral $ADEC$ is the difference $96 - 37.5 = 58.5$.

4. **Same Quotient and Remainder.** When each of the numbers 887, 1242, and 2023 is divided by the integer $d > 1$, the same remainder r occurs. Determine the positive integers d and r.

   Solution.

   We show that $d = 71$ and $r = 35$.

   We have the following identities:

$$\begin{aligned} 887 &= ad + r \\ 1242 &= bd + r \\ 2023 &= cd + r \end{aligned}$$

   For some positive integers a, b, and c.

   By subtracting the first equation from the second one and the second equation from the third one, we get $355 = (b - a)d$ and $781 = (c - b)d$. In particular, d is a common divisor of 355 and 781 (different from 1). From the factorizations: $355 = 5 \cdot 71$ and $781 = 11 \cdot 71$, we get that $d = 71$. By dividing the three given integers by 71, we get: $887 = 12 \cdot 71 + 35$, $1242 = 17 \cdot 71 + 35$, and $2023 = 28 \cdot 71 + 35$. Therefore $r = 35$.

5. **Exponential Equation.** Determine all real values of x satisfying the following equation:

$$4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}.$$

   Solution.

   The only solution is $x = \frac{3}{2}$.

   First of all, we check that $x = \frac{3}{2}$ is a solution. As a matter of fact, the left hand side of the equation becomes:

$$4^{3/2} - 3^{3/2-1/2} = 8 - 3 = 5$$

   while the right hand side becomes

$$3^{3/2+1/2} - 2^{3-1} = 9 - 4 = 5.$$

   Therefore, $x = \frac{3}{2}$ satisfies the equation.

Conversely, if x satisfies the equation then we have:

$$\begin{aligned}
2^{2x}+2^{2x-1} &= 3^{x+1/2}+3^{x-1/2};\\
2^{2x-1}(2+1) &= 3^{x-1/2}(3+1);\\
3\cdot 2^{2x-1} &= 2^2\cdot 3^{x-1/2};\\
2^{2x-3} &= 3^{x-3/2};\\
4^{x-3/2} &= 3^{x-3/2};\\
\left(\frac{4}{3}\right)^{x-3/2} &= 1.
\end{aligned}$$

Therefore, we must have $x-\frac{3}{2}=0$, that is to say $x=\frac{3}{2}$.

6. **Minimum Value.** Determine the minimum value of

$$\frac{x}{3y}+\frac{6y}{z}+\frac{4z}{x}$$

for positive real numbers x, y, z.

Solution.

The minimum value is 6. From the AM-GM inequality, we have:

$$\frac{x}{3y}+\frac{6y}{z}+\frac{4z}{x}\geq 3\sqrt[3]{\frac{x}{3y}\cdot\frac{6y}{z}\cdot\frac{4z}{x}}=3\sqrt[3]{8}=6.$$

The 6 value is achieved, for example, when $x=6$, $y=1$, and $z=3$.

Alternate Approach. Let y be a fixed real number and consider the function in two variables:

$$F(x,z):=\frac{x}{3y}+\frac{6y}{z}+\frac{4z}{x}.$$

By setting the gradient of $F(x,y)$ equal to zero we get:

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial x} &= \frac{1}{3y}-\frac{4z}{x^2}=0,\\
\frac{\partial F(x,y)}{\partial z} &= -\frac{6y}{z^2}+\frac{4}{x}=0.
\end{aligned}$$

From the second equation we get

$$x=\frac{2z^2}{3y}.$$

Then, by plugging in the first equation, we get:

$$\begin{aligned}
\frac{1}{3y}-\frac{9y^2}{z^3} &= 0,\\
z^3 &= 27y^3\\
z &= 3y.
\end{aligned}$$

This also implies that $x=6y$. A direct computation shows that $F(6y,3y)=6$. Moreover, the determinant of the Hessian matrix evaluated at $x=6y$ and $z=3y$:

$$\begin{vmatrix}\frac{1}{9y^2} & -\frac{1}{9y^2}\\ -\frac{1}{9y^2} & \frac{4}{9y^2}\end{vmatrix}=\frac{3}{9y^2}$$

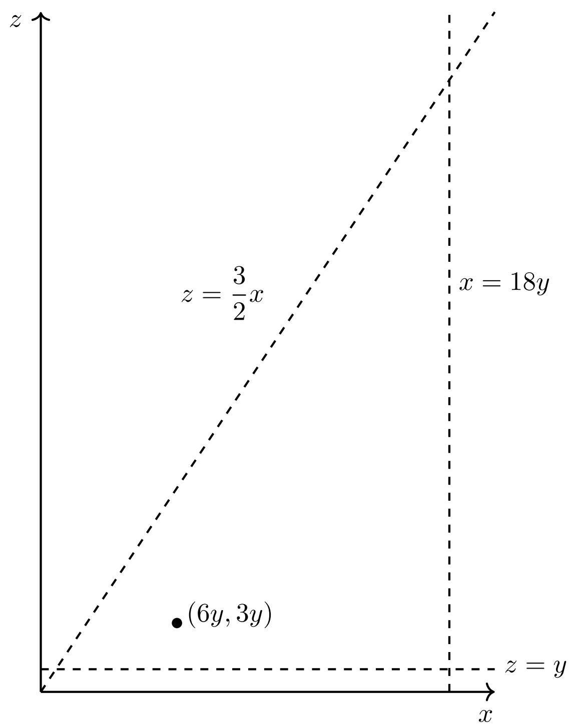
is positive and we also have $F_{xx}(6y,3y)=1/(9y^2)>0$. Therefore, the point $(6y,3y)$ is a local minimum. To prove that this is also a global minimum we need to study the end behavior of the function $F(x,z)$

on the region $x > 0$, $z > 0$. It is enough to show that the function $F(x, z)$ has values larger than 6 outside an open and bounded region containing the point $(6y, 3y)$.

Let us consider the following three lines on the x, z plane:

$$\begin{aligned} z &= y \\ z &= \frac{3}{2}y \\ x &= 18y \end{aligned}$$

The open and bonded region containing the point $(6y, 3y)$ we want to consider is the interior of the triangle defined by these three lines:



If $0 < z \leq y$, we have:

$$F(x, z) = \frac{x}{3y} + \frac{6y}{z} + \frac{4z}{x} > \frac{6y}{z} \geq 6.$$

If $x \geq 18y$, we have:

$$F(x, z) = \frac{x}{3y} + \frac{6y}{z} + \frac{4z}{x} > \frac{x}{3y} \geq 6.$$

If $z \geq \frac{3}{2}x$, we have:

$$F(x, z) = \frac{x}{3y} + \frac{6y}{z} + \frac{4z}{x} > \frac{4z}{x} \geq 6.$$

In conclusion, for every (x, z) outside of the interior of the triangle defined by the three dashed lines, we have $F(x, z) > 6$. Therefore, the point $(6y, 3y)$ is an absolute minimum and the minimum value is $F(6y, 3y) = 6$. Note that the value 6 does not depend on the fixed value of y so for positive real numbers x, y, z, the minimum value of

$$\frac{x}{3y} + \frac{6y}{z} + \frac{4z}{x}$$

is 6.

7. **Choose a Real Number.** Suppose that the real number x is chosen uniformly at random in the interval $(200, 300)$. Given that $\lfloor\sqrt{x}\rfloor = 15$, determine the probability that $\lfloor\sqrt{100x}\rfloor = 150$. Express your answer as a rational fraction in lowest terms.

   Note. For any real number a, the notation $\lfloor a\rfloor$ represents the largest integer less than or equal to a.

   Solution.

   The probability is $\dfrac{301}{3100}$.

   First of all, we have that $\lfloor\sqrt{x}\rfloor = 15$ if and only if $15 \le \sqrt{x} < 16$ that is to say, $225 \le x < 256$. Therefore, the total space has measure $256 - 225 = 31$.

   On the other hand, we have that $\lfloor\sqrt{100x}\rfloor = \lfloor 10\sqrt{x}\rfloor = 150$ if and only if

$$\begin{array}{rcl} 150 \le & 10\sqrt{x} & < 151 \\ 15 \le & \sqrt{x} & < 15.1 \\ 225 \le & x & < 228.01. \end{array}$$

   In particular, the measure of the space we are interested in is $228.01 - 225 = 3.01$.

   Therefore, the probability is $\dfrac{3.01}{31} = \dfrac{301}{3100}$.

8. **Roots of a Polynomial.** Let p, q, and r be distinct complex roots of the equation $x^3 - x^2 + x - 2 = 0$. Determine the value of $p^3 + q^3 + r^3$.

   Solution.

   We get $p^3 + q^3 + r^3 = 4$. Thanks to the Factor Theorem, we can write:

$$x^3 - x^2 + x - 2 = (x - p)(x - q)(x - r).$$

   By expanding the right side of the equation and equating the coefficients, we get:

$$\begin{aligned} p + q + r &= 1, \\ pq + pr + qr &= 1, \\ prq &= 2. \end{aligned}$$

   As a side note, from the fundamental theorem of symmetric forms, any symmetric form in p, q, and r (like $p^3 + q^3 + r^3$) can be written in terms of $p + q + r$, $pq + pr + qr$, and pqr, so, in theory, we can determine the value of $p^3+q^3+r^3$. However, in what follows, we will only need the identity $p+q+r = 1$.

   We consider the following identity:

$$\begin{aligned} (p + q + r)^3 &= p^3 + q^3 + r^3 + 3(q + r)(p + r)(p + q) \\ &= p^3 + q^3 + r^3 + 3(1 - p)(1 - q)(1 - r) \end{aligned}$$

   Now, the term $(1-p)(1-q)(1-r)$ is the evaluation of the polynomial $x^3-x^2+x-2 = (x-p)(x-q)(x-r)$ at $x = 1$, therefore, it is equal to -1. Therefore, we get:

$$1^3 = p^3 + q^3 + r^3 - 3,$$

   from which we get $p^3 + q^3 + r^3 = 4$.

9. **A Functional Equation.** Determine all functions on the real numbers satisfying

$$f(1-x)+2 = xf(x).$$

Solution.

There exists a unique solution:

$$f(x) = \frac{2x-4}{x^2-x+1}.$$

First, we check that this function satisfies the equation:

$$\begin{aligned} f(1-x)+2 &= \frac{2(1-x)-4}{(1-x)^2-(1-x)+1}+2 \\ &= \frac{-2x-2}{x^2-x+1}+2 \\ &= \frac{2x^2-4x}{x^2-x+1} \\ &= x\cdot\frac{2x-4}{x^2-x+1} \\ &= xf(x). \end{aligned}$$

On the other hand, by replacing x with $1-x$ in the equation:

$$f(1-x)+2 = xf(x)$$

we get:

$$f(x)+2 = (1-x)f(1-x)$$

From the first equation, we have $f(1-x) = xf(x)-2$. By substituting in the second equation, we obtain:

$$f(x)+2 = (1-x)(xf(x)-2).$$

Finally, we solve for $f(x)$:

$$\begin{aligned} f(x)+2 &= xf(x)-2-x^2f(x)+2x \\ (x^2-x+1)f(x) &= 2x-4 \\ f(x) &= \frac{2x-4}{x^2-x+1}. \end{aligned}$$

10. **Multiple Locks.** The door to a high security area is to have multiple locks. Each of the eleven persons is to receive an incomplete set of keys, subject to the following condition: Whenever six of the eleven are present, they have among them a key to every lock, but whenever fewer than six are present there is at least one of the locks that none of them can open. What is the smallest number of locks that will allow such a distribution of keys?

Solution.

The minimum number of locks is $\binom{11}{5} = 462$.

Let us define a function f from the set S of all five-person subsets to the set L of the locks. For each subset s_i of five people, we choose $f(s_i) = l_i$ where l_i is a lock such that no one of the persons in s_i can open.

Now, we claim that, no matter how we choose the locks $f(s_i)$, the function f is one-to-one. To show this, assume that $f(s) = f(q) = l$, that is to say, no one in s and no one in q can open the lock l. If the set s is different from the set q, then the union set $s \cup q$ would include at least 6 different people that cannot open lock l, contradicting the fact that whenever six of the eleven are present, they have among them a key to every lock. Therefore, the two sets s and q must be equal, and the function $f: S \to L$ is one-to-one. As a consequence, the number of elements of S must be less than or equal to the number of locks. Since there are $\binom{11}{5} = 462$ subsets of five people, we must have at least 462 locks.

To see that this number suffices, let $s_1, s_2, \ldots, s_{462}$ be the 462 five-persons subsets, and $l_1, l_2, \ldots, l_{462}$ be 462 different locks. Then give each of the eleven individuals a key to every lock except those l_i for which the individual is a member of s_i. This way, no one in set s_i will be able to open lock l_i. On the other hand, the only people who cannot open a specific lock l_i are the five persons in subset s_i. Therefore, any subset of 6 people will include one person that can open lock l_i. The same argument applies to every lock, therefore any subset of 6 people can open all the locks.