28th Annual Iowa Collegiate Mathematics Competition

Saturday, March 12, 2022

problems by Matthew Wright, St. Olaf College

To receive full credit, all solutions require complete justification.
Calculators are allowed but not very helpful and certainly not necessary.
Books, notes, and other resources are prohibited.

1. Root of the matter. Find all real numbers x that satisfy $x = x\sqrt{x} - 6\sqrt{x}$.

Solution: One solution is $x = 0$. If $x \neq 0$, we can divide both sides by $\sqrt{x}$ to obtain $\sqrt{x} = x - 6$, or equivalently $0 = x - \sqrt{x} - 6$. Let $y = \sqrt{x}$ and we have a quadratic:

$$0 = y^2 - y - 6$$

This factors as $0 = (y-3)(y+2)$, so $y = 3$ or $y = -2$. If $y = 3$ then $x = 9$. However $y = -2$ gives no solution for x since $\sqrt{x} \geq 0$.

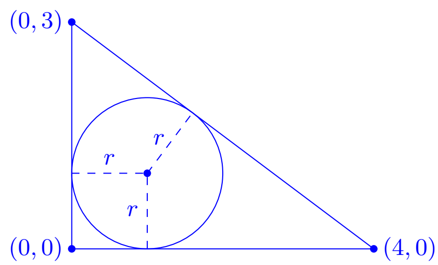
Therefore, the solutions are $x = 0$ and $x = 9$.

2. Fourth-degree difference. Show that there are no integers a and b that satisfy $a^4 - b^4 = 2022$.

Solution: Work modulo 5. The fourth powers mod 5 are either 0 or 1. So $a^4 - b^4$ must be congruent to 0, 1, or 4 modulo 5. Since $2022 \equiv 2 \pmod 5$, there are no integer solutions.

3. Inscribed circle. Triangle T has side lengths 3, 4, and 5. A circle of radius r is inscribed in T (tangent to all three sides). What is the radius r?

Solution:



Then the center of the inscribed circle is (r, r). The equation of the hypotenuse is $y = -\frac{3}{4}x + 3$, or equivalently $3x + 4y - 12 = 0$. We must find r such that the distance from (r, r) to $3x + 4y - 12 = 0$ is r. This distance is given by

$$\frac{|3r + 4r - 12|}{\sqrt{3^2 + 4^2}} = \frac{|7r - 12|}{5} = r.$$

Solving this equation for r, we find $r = 1$ or $r = 6$. The solution $r = 1$ gives the inscribed circle.

Alternatively, one can solve this problem without knowing the formula for the distance from a point to a line. For example, draw a line through (r, r) perpendicular to the hypotenuse, and find that this line intersects the hypotenuse at $\left(\frac{36+4r}{25}, \frac{48-3r}{25}\right)$. The distance from the center of the circle to this intersection point must be r, which leads to the solution $r = 1$.

4. Sumthing to consider. Evaluate $\displaystyle\sum_{n=2}^{\infty} \frac{n+3}{n^3-n}$ exactly.

Solution: First apply partial fractions:

$$\frac{n+3}{n^3-n} = \frac{2}{n-1} - \frac{3}{n} + \frac{1}{n+1}$$

The sum is then:

$$\sum_{n=2}^{\infty}\left(\frac{2}{n-1} - \frac{3}{n} + \frac{1}{n+1}\right) = \left(\frac{2}{1} - \frac{3}{2} + \frac{1}{3}\right) + \left(\frac{2}{2} - \frac{3}{3} + \frac{1}{4}\right) + \left(\frac{2}{3} - \frac{3}{4} + \frac{1}{5}\right) + \left(\frac{2}{4} - \frac{3}{5} + \frac{1}{6}\right) + \cdots$$

Observe that this sum telescopes, in the sense that each middle term after the first cancels with terms to its left and right that share the same denominator.

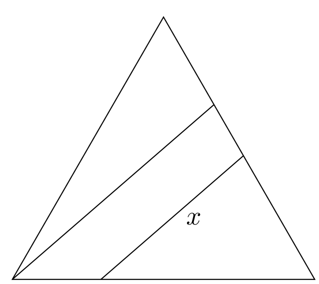
$$\left(\frac{2}{1} - \frac{3}{2} + \cancel{\frac{1}{3}}\right) + \left(\frac{2}{2} - \cancel{\frac{3}{3}} + \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{3}{4}} + \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{2}{4}} - \cancel{\frac{3}{5}} + \cancel{\frac{1}{6}}\right) + \cdots$$

The only terms that don't cancel are:

$$\frac{2}{1} - \frac{3}{2} + \frac{2}{2} = \frac{3}{2}.$$

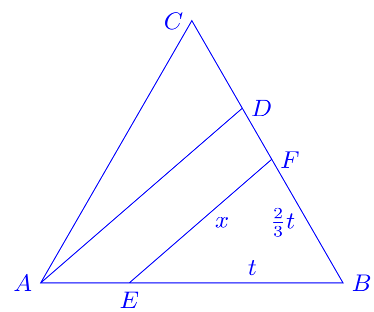
Therefore $\displaystyle\sum_{n=2}^{\infty} \frac{n+3}{n^3-n} = \frac{3}{2}$.

5. Three equal areas. The following diagram shows an equilateral triangle with side length 1. Parallel line segments, one passing through a vertex of the triangle, divide the triangle into three regions of equal area. What is the length x of the shorter line segment? (Recall that the area of a triangle can be expressed as $\frac{1}{2}ab\sin(\theta)$, where a and b are the lengths of two sides and θ is the angle between them.)



Solution: The area of an equilateral triangle with side length 1 is $\frac{\sqrt{3}}{4}$. Thus, each of the three regions has area $\frac{\sqrt{3}}{12}$.

Label vertices and intersection points as in the diagram below. Since the area of ΔADC is $\frac{1}{3}$ the area of ΔABC, length $CD = \frac{1}{3}$. Thus, length $BD = \frac{2}{3}$.



Let t be the length EB. Since $\frac{BD}{AB} = \frac{2}{3}$ and ΔABD is similar to ΔEBF, length $BF = \frac{2}{3}t$.

We can express the area of ΔEBF as the half the product of two side lengths and the sine of the included angle:

$$\frac{1}{2}t\left(\frac{2}{3}t\right)\sin(60^\circ) = \frac{\sqrt{3}}{12}$$

Solving the equation above, we find $t = \frac{1}{\sqrt{2}}$.

We can also apply the law of cosines:

$$\begin{aligned} x^2 &= t^2 + \frac{4}{9}t^2 - 2\cdot\frac{2}{3}t^2\cos(60^\circ) \\ &= \frac{1}{2} + \frac{4}{9}\left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{2}{9} - \frac{1}{3} \\ &= \frac{7}{18} \end{aligned}$$

Therefore, $x = \sqrt{\frac{7}{18}} = \frac{1}{3}\sqrt{\frac{7}{2}} \approx 0.6236$.

6. Another sumthing. Evaluate $\displaystyle\sum_{j=1}^{\infty} \frac{1}{2^j}\ln\left(\frac{2^j}{3}\right)$. Express your answer as a logarithm.

Solution: First, observe that we can apply log rules to write the sum as two simpler sums:

$$\sum_{k=1}^{\infty} \frac{1}{2^k}\ln\left(\frac{2^k}{3}\right) = \ln 2\sum_{k=1}^{\infty}\frac{k}{2^k} - \ln 3\sum_{k=1}^{\infty}\frac{1}{2^k}$$

The second sum above is a geometric series, which sums to 1.

For the first sum, recall a more general geometric series from calculus class:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for} \quad |x| < 1.$$

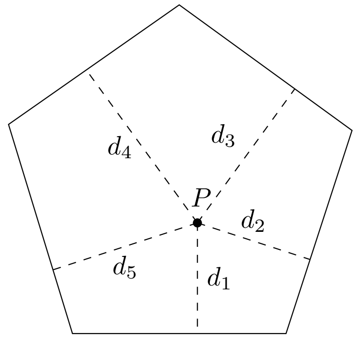
This formula can be differentiated as

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{for} \quad |x| < 1.$$

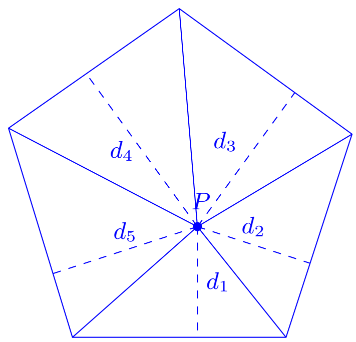
Setting $x = \frac{1}{2}$, we find $\sum_{k=1}^{\infty}\frac{k}{2^k} = 2$. Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{2^k}\ln\left(\frac{2^k}{3}\right) = 2\ln 2 - \ln 3 = \ln\frac{4}{3}.$$

7. A point in a pentagon. Point P is located inside of a regular pentagon, and d_1, d_2, d_3, d_4, d_5 are the (perpendicular) distances from P to each of the sides of the pentagon. Show that $d_1 + d_2 + d_3 + d_4 + d_5$ is constant regardless of where P is in the pentagon.



Solution: Draw segments from P to each vertex of the pentagon, partitioning the pentagon into five triangles as shown:



Let s be the side length of the pentagon. Then write the area A of the pentagon as a sum of the areas of the triangles:

$$A = \frac{sd_1}{2} + \frac{sd_2}{2} + \frac{sd_3}{2} + \frac{sd_4}{2} + \frac{sd_5}{2}$$

Thus

$$\frac{2A}{s} = d_1 + d_2 + d_3 + d_4 + d_5.$$

This shows that the sum of the five distances does not depend on the location of P in the pentagon.

The line above completes the solution. However, since $A = \frac{s^2}{4}\sqrt{25 + 10\sqrt{2}}$, we can express also the result as

$$d_1 + d_2 + d_3 + d_4 + d_5 = \frac{s}{2}\sqrt{25 + 10\sqrt{2}} \approx 3.44095s.$$

8. An integral game. Consider the following game played on points with integer coordinates in the plane. You start at $(0,0)$ and make a sequence of moves. The possible moves are:

(a) Move diagonally. That is, you may move from (x,y) to $(x+1,y+1)$, $(x+1,y-1)$, $(x-1,y+1)$, or $(x-1,y-1)$.

(b) Square both coordinates. That is, you may move from (x,y) to (x^2,y^2).

(c) Negate both coordinates. That is, you may move from (x,y) to $(-x,-y)$.

You win the game when you move to $(1,0)$. Either give a sequence of moves that win the game or prove that winning is not possible.

Solution: Winning is not possible. Moreover, it is not possible to move to any point (x,y) with $x+y \equiv 1 \pmod 2$.

To see this, observe that each possible move from (x,y) preserves the parity of $x+y$. Since the game starts at $(0,0)$, every move will be to a point (x,y) with $x+y \equiv 0 \pmod 2$.

9. Three random points. Three points are selected uniformly at random from the closed interval $[0,10]$. What is the probability that all three lie within a subinterval of length 2?

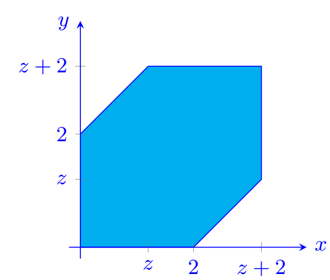
Solution: Let random variables X, Y, and Z be the locations of the three points.

Let E be the event that all three points lie within a subinterval of length 2. That is, E is the event that

$$|X-Y| \le 2, \qquad |X-Z| \le 2, \qquad \text{and} \qquad |Y-Z| \le 2.$$

Fix a value z of Z. Then consider the conditional probability $P(E \mid Z=z)$.

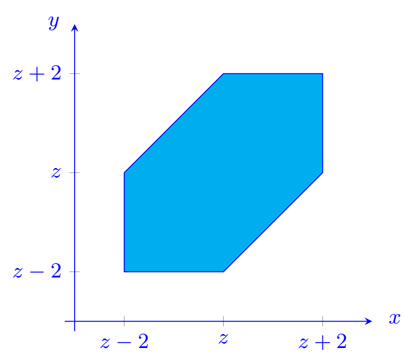
If $0 \le z \le 2$, then E occurs if the values x and y of X and Y (respectively) satisfy $0 \le x \le z+2$, $0 \le y \le z+2$, and $|x-y| \le 2$. In other words, E occurs if the point (x,y) lies in the shaded region below:



The area of the shaded region is $(z+2)^2 - z^2 = 4z+4$. Since the joint density of X and Y is $\frac{1}{100}$ on $[0,10]^2$, we have

$$P(E \mid Z = z \in [0,2]) = \frac{4z+4}{100}.$$

Similarly, if $2 < z < 8$, then E occurs if (x,y) lies in the following shaded region:



The area of this shaded region is 12, so we have

$$P(E \mid Z = z \in [2,8]) = \frac{12}{100}.$$

The case $8 \le z \le 10$ is symmetric to the case $0 \le z \le 2$. Specifically, we replace z with $10 - z$ and obtain

$$P(E \mid Z = z \in [8,10]) = \frac{44 - 4z}{100}.$$

We can now express the conditional probability of E given $Z = z$ in piecewise form:

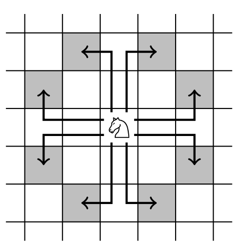
$$P(E \mid Z = z) = \begin{cases} \frac{4z+4}{100} & \text{if } 0 \le z \le 2, \\ \frac{12}{100} & \text{if } 2 < z < 8, \\ \frac{44-4z}{100} & \text{if } 8 \le z \le 10. \end{cases}$$

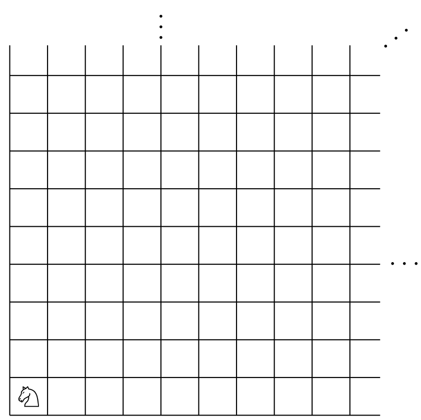
The (marginal) density of Z is $\frac{1}{10}$ on $[0, 10]$. Thus, we compute the probability of E as:

$$\begin{aligned} P(E) &= \int_0^{10} \frac{1}{10} P(E \mid Z = z) dz \\ &= \int_0^2 \frac{4z+4}{1000} dz + \int_2^8 \frac{12}{1000} dz + \int_8^{10} \frac{44-4z}{1000} dz \\ &= \frac{1}{1000} \left(\int_0^2 (4z+4)\, dz + \int_2^8 12\, dz + \int_8^{10} (44 - 4z)\, dz \right) \\ &= \frac{1}{1000} (16 + 72 + 16) \\ &= \frac{104}{1000} = \frac{13}{125}. \end{aligned}$$

Thus, $P(E) = \frac{13}{125} = 0.104$.

10. Knight moves. Recall that a knight can move on a chessboard in an "L" shape, as shown below left. Suppose that a knight starts at the lower-left corner of a chessboard that extends infinitely to the right and up, as shown below right. How many squares are accessible to the knight in 100 moves but not in 99 or fewer moves?

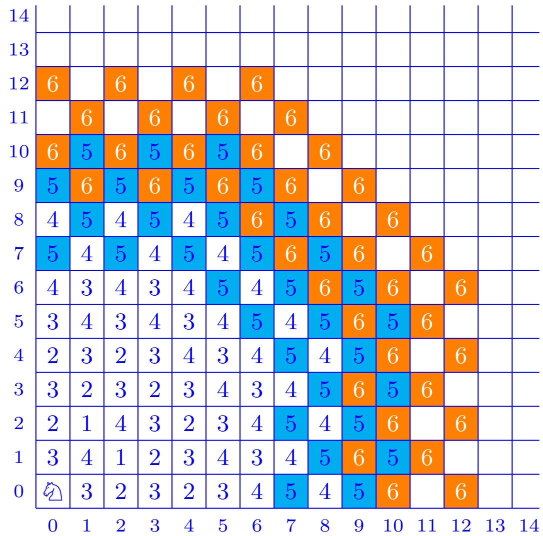




Solution: The number of squares accessible to the knight in m moves but not in $m - 1$ moves is:

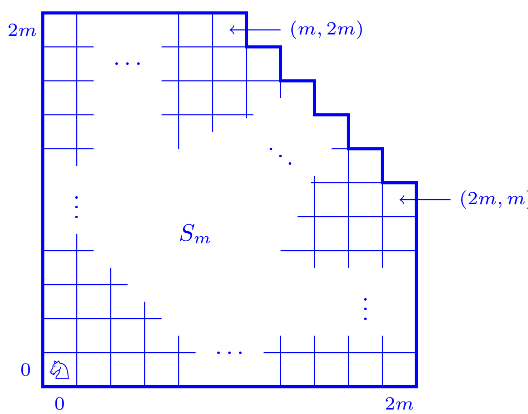
moves	number of squares
1	2
2	9
3	20
4	27
$m \ge 5$	$7m - 3$

To justify the values in the above, consider the diagram below. In the diagram, the numbers in the squares indicate the minimum number of moves it takes the knight to reach each square (up to 6 moves). It takes four moves to access all of the squares adjacent to the knight's starting position. The pattern in newly-accessible squares becomes more clear at move 5, as shown by the shaded squares.



Index the squares with horizontal and vertical coordinates as labeled in the margins of the diagram above. Let (i, j) refer to the square with horizontal index i and vertical index j.

For $m \geq 5$, let S_m be the set of squares (i, j) satisfying $0 \leq i \leq 2m$, $0 \leq j \leq 2m$, and $i + j \leq 3m$. The following diagram illustrates S_m.



Define the *parity* of a square (i, j) to be *even* if $i + j$ is even, and *odd* if $i + j$ is odd. Note that each move takes the knight from a square with a given parity to a square with the opposite parity.

For $m \geq 5$, the following properties hold:

(a) All squares in S_m with the same parity as m are accessible in m moves or fewer. (Some squares in S_m with opposite parity from m are not accessible in m moves.)

(b) All squares in S_m are accessible in $m + 1$ moves or fewer.

These two properties follow by induction on m, starting with the $m = 5$ base case.

Thus, the set of squares accessible to the knight after m or fewer moves consists of the squares in S_m with the same parity as m, together with the squares in S_{m-1} with the same parity as $m - 1$. (For example, in up to 6 moves the knight may access any of the even squares in S_6 and any of the odd squares in S_5.)

Let $Q(m)$ be the number of squares in S_m with the same parity as m. The number of squares accessible in m moves or fewer is $Q(m) + Q(m - 1)$.

We next determine $Q(m)$, considering even and odd m separately:

- **If m is even:** The number of even squares (i,j) with $0 \le i \le 2m$ and $0 \le j \le 2m$ is $2m^2+2m+1$. However, $\frac{m^2}{4}$ of these squares are such that $i+j > 3m$. Thus

$$Q(m) = 2m^2 + 2m + 1 - \frac{m^2}{4} = \frac{7m^2 + 8m + 4}{4} \qquad \text{if } m \text{ is even.}$$

- **If m is odd:** The number of odd squares (i,j) with $0 \le i \le 2m$ and $0 \le j \le 2m$ is $2m^2+2m$. However, $\frac{m^2-1}{4}$ of these squares are such that $i+j > 3m$. Thus

$$Q(m) = 2m^2 + 2m - \frac{m^2-1}{4} = \frac{7m^2 + 8m + 1}{4} \qquad \text{if } m \text{ is odd.}$$

Let $N(m)$ be the number of squares accessible in m moves or fewer. We compute $N(m)$:

$$N(m) = Q(m) + Q(m-1) = \frac{7m^2 + m + 2}{2}$$

and this does not depend on the parity of m.

Thus the number of squares accessible in m moves but not fewer is

$$N(m) - N(m-1) = \frac{7m^2 + m + 2}{2} - \frac{7(m-1)^2 + (m-1) + 2}{2} = 7m - 3.$$

Therefore the number of squares accessible in 100 moves but not in 99 moves is $7(100) - 3 = 697$.