28th Annual Iowa Collegiate Mathematics Competition

Saturday, March 12, 2022

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To receive full credit, all solutions require complete justification.
Calculators are allowed but not very helpful and certainly not necessary.
Books, notes, and other resources are prohibited.

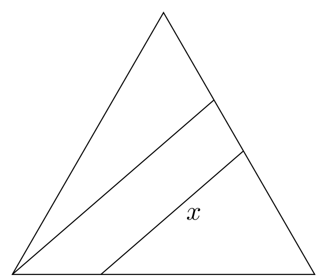
1. **Root of the matter.** Find all real numbers x that satisfy $x = x\sqrt{x} - 6\sqrt{x}$.

2. **Fourth-degree difference.** Show that there are no integers a and b that satisfy $a^4 - b^4 = 2022$.

3. **Inscribed circle.** Triangle T has side lengths 3, 4, and 5. A circle of radius r is inscribed in T (tangent to all three sides). What is the radius r?

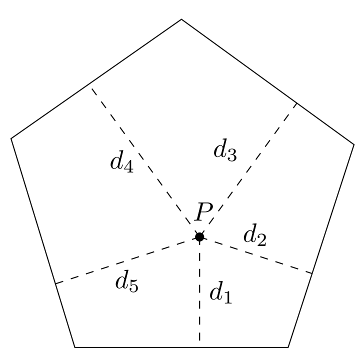
4. **Sumthing to consider.** Evaluate $\displaystyle\sum_{n=2}^{\infty} \frac{n+3}{n^3-n}$ exactly.

5. **Three equal areas.** The following diagram shows an equilateral triangle with side length 1. Parallel line segments, one passing through a vertex of the triangle, divide the triangle into three regions of equal area. What is the length x of the shorter line segment? (Recall that the area of a triangle can be expressed as $\frac{1}{2}ab\sin(\theta)$, where a and b are the lengths of two sides and θ is the angle between them.)



6. **Another sumthing.** Evaluate $\displaystyle\sum_{j=1}^{\infty} \frac{1}{2^j} \ln\left(\frac{2^j}{3}\right)$. Express your answer as a logarithm.

7. **A point in a pentagon.** Point P is located inside of a regular pentagon, and d_1, d_2, d_3, d_4, d_5 are the (perpendicular) distances from P to each of the sides of the pentagon. Show that $d_1 + d_2 + d_3 + d_4 + d_5$ is constant regardless of where P is in the pentagon.



8. An integral game. Consider the following game played on points with integer coordinates in the plane. You start at $(0,0)$ and make a sequence of moves. The possible moves are:

(a) Move diagonally. That is, you may move from (x,y) to $(x+1,y+1)$, $(x+1,y-1)$, $(x-1,y+1)$, or $(x-1,y-1)$.

(b) Square both coordinates. That is, you may move from (x,y) to (x^2,y^2).

(c) Negate both coordinates. That is, you may move from (x,y) to $(-x,-y)$.

You win the game when you move to $(1,0)$. Either give a sequence of moves that win the game or prove that winning is not possible.

9. Three random points. Three points are selected uniformly at random from the closed interval $[0,10]$. What is the probability that all three lie within a subinterval of length 2?

10. Knight moves. Recall that a knight can move on a chessboard in an "L" shape, as shown below left. Suppose that a knight starts at the lower-left corner of a chessboard that extends infinitely to the right and up, as shown below right. How many squares are accessible to the knight in 100 moves but not in 99 or fewer moves?