26th Iowa Collegiate Mathematics Competition, Solutions

Problems and Solutions by Stan Wagon, Macalester College, St. Paul, Minnesota

1. Fuel Economy. Alice drives a truck that gets 10 miles per gallon. Bob's car gets 40 mpg. Over a year they drive the same distance. Alice trades in her truck for a 14 mpg model. Bob wants to trade his car for one getting X mpg, such that his fuel savings over one year will match Alice's. What should X be?

Solution. No X will work; even a perpetual motion machine, for which $X = \infty$, would not save Bob as much fuel as Alice saves. For M miles, Alice's fuel saving is $\frac{M}{10} - \frac{M}{14}$, or $\frac{M}{35}$ gallons. Bob's saving is $\frac{M}{40} - \frac{M}{X}$. But this is less than $\frac{M}{40}$ which is less than $\frac{M}{35}$. For example, in 35 miles Alice saves one gallon. Bob now expends $35/40$, or $7/8$ of a gallon for that distance, and so cannot save a gallon. *More.* Various other aspects of the fuel economy paradox and a fuel economy singularity are explained in S. Wagon, Resolving the fuel economy singularity, *Math Horizons*, 26:1, Sept. 2018, 5–9.

2. Tennis Anyone? Alice, Bob, and Charlie play tennis in sets. Two of them, chosen randomly, play a set and the winner stays on the court for the next set, with the loser replaced by the idle player. At the end of the day, Alice played 15 sets, Bob played 14 sets, and Charlie played 9 sets. Who played in the 13th set?

Solution. The total number of sets is half of $15 + 14 + 9 = 38$, so 19. The only way Charlie plays only 9 is if he plays in each even-numbered set. Therefore Alice and Bob played in the 13th set.

3. Apologies to Ramanujan. The number 1729 has the property that it is divisible by the sum of its digits. In fact, 1728 has this property as well. Find three consecutive numbers larger than 1729 such that each is divisible by the sum of its decimal digits.

Solution. Using many 0 digits makes for simple sums. So use 10010, 10011, 10012.

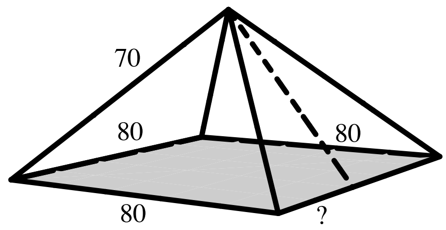
4. Which Integers are Dots? Let $P(k)$ be the assertion: Every integer is a dot product of two arithmetic progressions of integers, each of length k. For example, $P(3)$ is true if and only if every integer n has the form $a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\{a_i\}$ and $\{b_i\}$ are arithmetic progressions. Determine the truth or falsity of $P(3)$ and $P(4)$.

Note. An *arithmetic progression* is any sequence of the form $a, a+b, a+2b, a+3b, \ldots$, where a and b are real numbers. The *dot product* of two vectors is $(x, y, z, \ldots) \cdot (X, Y, Z, \ldots) = xX + yY + zZ + \ldots$.

Solution. For $P(4)$ suppose the two APs are $(a, a+d, a+2d, a+3d)$ and $(b, b+c, b+2c, b+3c)$. The dot product expands to $4ab + 6ac + 6bd + 14cd$, which is even and cannot represent 1.

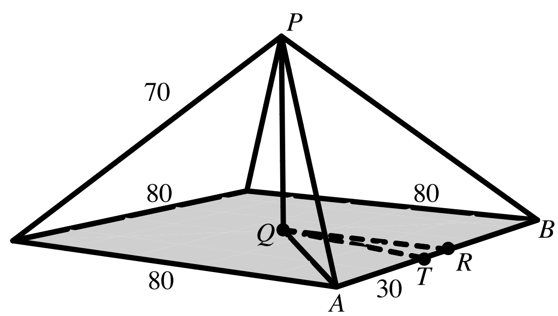
But $P(3)$ is true. Consider two general APs: $a + ic$ and $b + id$, where $i = 0, 1, 2$. The dot product is $3ab + 3bc + 3ad + 5cd$. Try $a = 0$ and $c = 1$ for the first AP: This then gives $3b + 5d$. Setting $b = \beta n$ and $d = \delta n$, we are searching for $n \cdot (3\beta + 5\delta) = n$. But $\gcd(3, 5) = 1$, so we can solve $3\beta + 5\delta = 1$ finding $\beta = 2$ and $\delta = -1$. So $b = 2n$ and $c = -n$ works: $(0, 1, 2) \cdot (2n, n, 0) = n$. *More.* Further analysis shows that the only true cases are $P(1)$, $P(2)$, $P(3)$, and $P(6)$.

5. Climb the Pyramid You're standing next to a pyramid whose base is a square, 80 feet on a side. The distance from each corner of the base to the top of the pyramid is 70 feet. You want to start at some point along the base of the pyramid and walk up one of the faces in a straight line to the top such that your path makes a 45° angle with the ground. How far from a corner of the base should you start? In the figure, the path you seek is shown as a dashed line.



Solution. Start 30 feet from a corner. Let $ABCD$ be the square base of the pyramid and P the top. Let Q be the point on the base directly below P. Then AQP is a right triangle with $AQ = 40\sqrt{2}$ and $AP = 70$, so

$PQ = \sqrt{70^2 - (40\sqrt{2})^2} = \sqrt{1700} = 10\sqrt{17}$. So you should start at a point T at the base of the pyramid such that $TQ = PQ = 10\sqrt{17}$. Let R be the midpoint of AB and suppose T is between A and R. Then $\angle TRQ = 90°$, so $TR = \sqrt{(10\sqrt{17})^2 - 40^2} = \sqrt{100} = 10$, and therefore $AT = 40 - 10 = 30$.



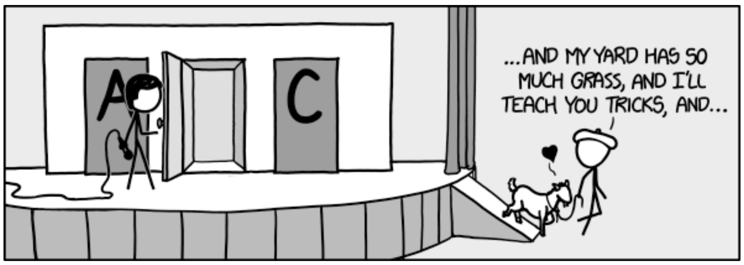
6. A Car and a Goat. Behind three doors labeled 1, 2, 3 are randomly placed a car, a key, and a goat. Alice and Bob win the car if Alice finds the car and Bob finds the key. First Alice (with Bob not present) opens a door; if the car is not behind it she can open a second door. If she fails to find the car, they lose. If she does find the car, then all doors are closed and Bob gets to open one of the three doors in the hope of finding the key and, if not, trying again with a second door. Alice and Bob do not communicate except to make a plan beforehand. What is their best strategy?

Solution. Alice's chance of finding the car is $2/3$, so they cannot do better than that. Here is a strategy that succeeds with probability $2/3$, and is therefore best possible. Alice starts with door 1. If she sees a car then she is finished. If she sees a key, then she opens door 2; if instead door 1 hides a goat, then she opens door 3. Bob's then starts with door 2. If he finds the key, he is finished. Otherwise door 2 hides either a car, in which case he opens door 1, or a goat, in which case he opens door 3.

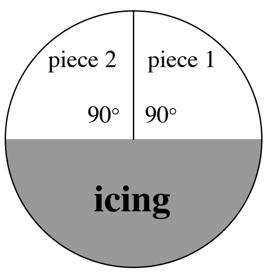
Assume Alice has found the car (otherwise they have lost). Suppose first that Alice finds it behind door 1. She is then done and Bob would open doors 2 and 3, guaranteeing that he finds the key. If Alice finds the key behind door 1, she opens door 2. Bob starts with door 2, behind which he sees the car and he would then open door 1, finding the key. Finally, if Alice saw the goat behind door 1, then she found the car behind door 3. Bob then sees the key behind door 2, and they succeeded with him opening only one door.

More. One can understand this more deeply from a permutation point of view. Consider the car as being tagged with number 1, the key with number 2, and the goat with 3. The number can be viewed as the item's home door. The arrangement of the items corresponds to a permutation P of 1 2 3; for example, if the order of the items is (goat, key, car), then the permutation is 3 2 1, which is the 2-cycle $3 \leftrightarrow 1$. The strategy just described is a pointer-following strategy: Alice starts by opening the home door of the car (door 1). If the car is not at home, she next opens the home door of the object she finds. Bob operates similarly, starting with the key's home door (2), and then, if necessary, moving to the door corresponding to what he found. If P is not a 3-cycle, then Alice will find the car because her two openings take care of the cases that the car lies in a 2-cycle or 1-cycle. And the same is true for Bob. And if P is a 3-cycle, then Alice (also Bob) will fail to find what they seek. Therefore the chance of success is $2/3$, because four of the six permutations are not 3-cycles. Note that in every case that Alice finds the car, Bob locates the key.

Underlying this sort of problem is the assumption that the contestants prefer the goat. Here is a nice take on this point by famed cartoonist Randall Munroe (known as xkcd; see ⟨https://xkcd.com/1282⟩).



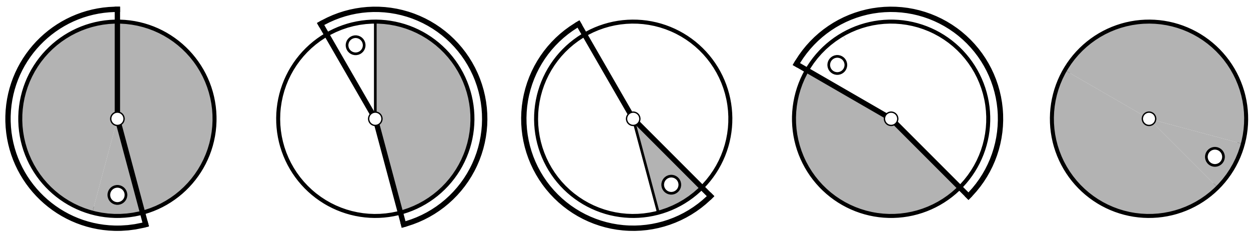
7. The Icing on the Cake. A round cake has icing on the top but not the bottom. Cut out a piece in the usual shape (a sector of a circle with vertex at the center), remove it, turn it upside down, and replace it in the cake to restore roundness. Do the same with the next piece; i.e., take a piece with the same vertex angle, but rotated counterclockwise from the first one so that one boundary edge coincides with a boundary edge of the first piece. Remove it, turn it upside-down, and replace it. The cake is self-healing: the cuts disappear once a piece is replaced. Keep doing this in a counterclockwise direction.



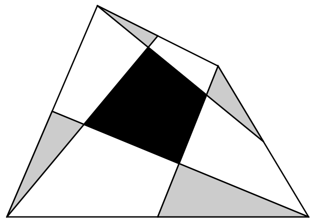
The figure shows the situation after two steps when the central angle is 90°.

If θ is the central angle of the pieces, let $f(\theta)$ be the smallest number of steps needed so that, under the repeated cutting-and-flipping just described, all icing returns to the top of the cake. For example, $f(90°) = 8$. What is $f(190°)$?

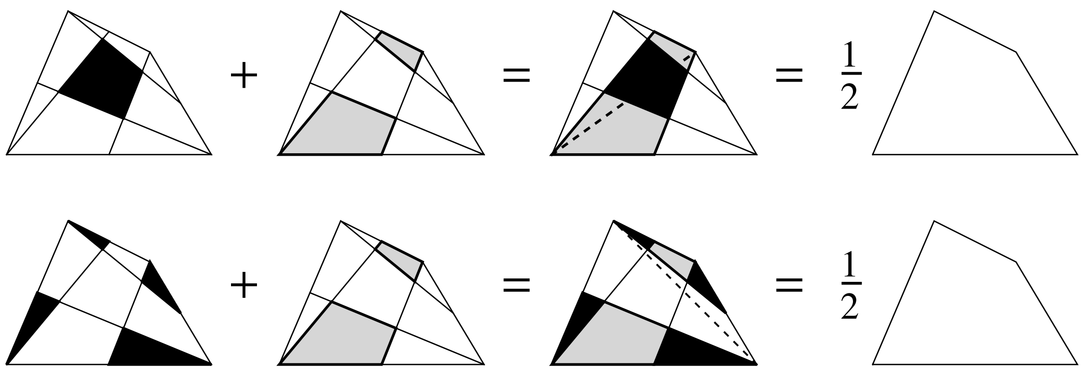
Solution. The icing returns to the top in four moves, as shown in the figure. The large white dot shows the path of a point during the process. *More.* The answer is 4 whenever $\pi < \theta < 2\pi$, even irrational angles θ such as 6 radians. The formula for all θ is tricky to work out: if $2\pi/\theta$ is an integer n, then $f(\theta) = 2n$; otherwise $f(\theta) = 2n(n+1)$, where $n = \lfloor 2\pi/\theta \rfloor$. For example, $f(1) = 2 \cdot 6 \cdot 7 = 84$.



8. A Crosscut Quadrilateral. Bisect the four sides of a convex quadrilateral and connect each midpoint to an opposite corner, choosing the first one in counterclockwise order as in the figure. Prove that the central quadrilateral has the same area as the union of the four corner triangles.



Solution. Let R be the central quadrilateral, let S be the union of the four corner gray regions, and let T be the union of the top and bottom white quadrilaterals. The figure shows that $R \cup T$ and $S \cup T$ each occupy one half of total area. In each case the dashed line divides the quadrilateral into two triangles and within each one the shaded and white sections have equal bases and the same altitude and so have the same area. Therefore area(R) = area(S).



9. Special Numbers. A set A of distinct positive integers is *special above n* if every $x \in A$ such that $x > n$ satisfies:

(1) x divides the product of all $y \in A$ with $y < x$; and (2) x does not divide any $y \in A$ with $y > x$.

For example, {1, 2, 3, 6, 9} is special above 3 and {1, 2, 3, 4, 12} is special above 4 but not special above 3. Find a set of size 9 that is special above 4. Partial credit will be given for answers having fewer than 9 numbers.

Solution. One answer is {1, 2, 3, 4, 24, 32, 36, 54, 81}. There are 40 examples of size 9, shown below. They all start with 1, 2, 3, 4, 24.

32 36 54 81	32 36 54 243	32 36 54 729	32 36 54 2187	32 36 162 243
32 36 162 729	32 36 162 2187	32 36 162 6561	36 54 64 81	36 54 64 243
36 54 64 729	36 54 64 2187	36 54 81 128	36 54 81 256	36 54 81 512
36 54 128 243	36 54 128 729	36 54 128 2187	36 54 243 256	36 54 243 512
36 54 256 729	36 54 256 2187	36 54 512 729	36 54 512 2187	36 64 162 243
36 64 162 729	36 64 162 2187	36 64 162 6561	36 128 162 243	36 128 162 729
36 128 162 2187	36 128 162 6561	36 162 243 256	36 162 243 512	36 162 256 729
36 162 256 2187	36 162 256 6561	36 162 512 729	36 162 512 2187	36 162 512 6561

A greedy algorithm will find {1, 2, 3, 4, 24, 32, 36, 54, 81}, but there is one delicate step. It is natural to start with 1, 2, 3, 4. This forces any number in the set to have the form 2^a3^b. We next need a divisor of 24, so the choices are 6, 8, 12, or 24. Choosing 6 at this step is a poor choice, because 6 divides almost all 2^a3^b; this leads to a dead end after 1, 2, 3, 4, 6, 8, 9 (or similar). Choosing 24 works out well, but from this point on we take the *smallest* number that works. So next comes a number that is larger than 24, divides

$1 \cdot 2 \cdot 3 \cdot 4 \cdot 24 = 24^2 = 576 = 2^6 3^2$, but is not a multiple of 24. The smallest such is 32. Next comes a number that is larger than 32, divides $2^{11}\,3^2$, and is not a multiple of 24 or 32. The smallest such is $2^2\,3^2 = 36$. Next comes a number larger than 36, dividing $2^{13}\,3^4$, and not a multiple of 24, 32, or 36. That is $54 = 2 \cdot 3^3$. And to finish we choose 81, the smallest divisor of $2^{14}\,3^7$ that works.

More. Let $f(n)$ be the size of the largest special above n set. One can prove that $f(n)$ is always finite. It is not hard to see that $f(1) = 1$, $f(2) = 2$, $f(3) = 5$, and $f(4) = 9$. But this does not prepare us for the enormous jump that comes next: $f(5) \geq 10^{10^{2.5\text{ billion}}}$. See ⟨https://oeis.org/A323457⟩ or the forthcoming problem book, *Bicycle or Unicycle?* by Dan Velleman and Stan Wagon (MAA Press).

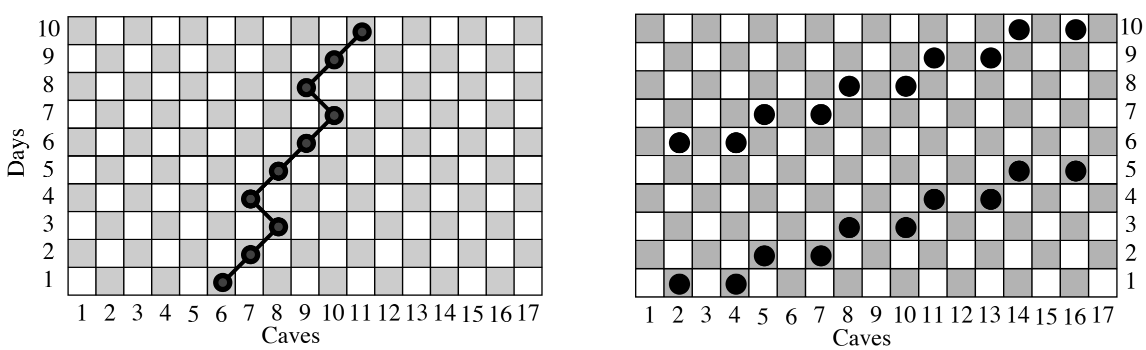
10. Where's Bob? Federal agent Alice is searching for computer hacker Bob, who is hiding in one of 17 caves. The caves form a linear array and every night Bob moves from the cave he is in to one of the caves on either side of it. Alice can search two caves each day, with no restrictions on her choice. For example, if Alice searches (1 2), (2 3), …, (16 17), then she is certain to catch Bob, though it might take 16 days.

Find a strategy by which Alice is guaranteed to find Bob in 10 days. Partial credit will be given for strategies that require 11, 12, 13, 14, or 15 days.

Solution. The following search strategy works in 10 days:

Day	1	2	3	4	5	6	7	8	9	10
Caves	2	5	8	11	14	2	5	8	11	14
Caves	4	7	10	13	16	4	7	10	13	16

The key is that if Bob is in an even-numbered cave, then he must be in an odd-numbered cave on the next day, and vice versa. A geometric view is given by a checkerboard pattern. In the left figure, Bob starts in an even cave, 6, and so he starts in a black square on the bottom row and he must always be in a black cave after that.



So the 10-day strategy given above can be discovered as in the right figure, where the disks are where Alice searches. Every path representing Bob's moves from Day 1 to Day 10 that starts on a black square will strike a disk by the fifth day; white paths, lead to capture in the last five days.

Many of these problems are discussed in greater detail in the soon-to-be-published book: *Bicycle or Unicycle? A Collection of Intriguing Mathematical Puzzles,* by Dan Velleman and Stan Wagon, MAA Press (Amer. Math. Soc.), Spring 2020.