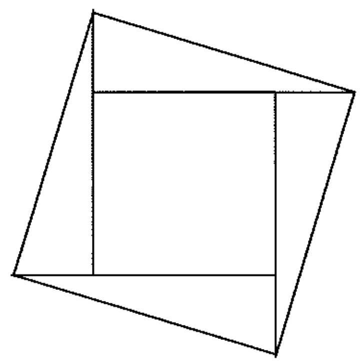
25th Annual Iowa Collegiate Mathematics Competition
Drake University, Saturday, February 23, 2019
Problems by Razvan Gelca

SOLUTIONS

1. Prove that there is no integer that when divided by 6 gives 2 as remainder, and when divided by 15 gives 12 as remainder.

Solution: A number that when divided by 15 gives 12 as remainder is of the form $15n+12 = 3(5n+4)$, which is a multiple of 3. A number that when divided by 6 gives 2 as remainder is of the form $3(2m)+2$, which is not a multiple of 3. So if such an integer existed, it would and would not be a multiple of 3 at the same time, which is impossible.

2. Four equal right triangles are arranged as in the figure below to form two squares, one inside the other. It is known that the exterior square has side equal to 41, and the interior square has side equal to 31. Find the sides of the right triangles.



Solution: Let the legs of the right triangles be x, y, with $x < y$. Then the area of the big square is equal to $x^2 + y^2$, while the area of the small square is $(x-y)^2$. We have to solve the system

$$x^2 + y^2 = 41^2$$
$$(x-y)^2 = 31^2.$$

We have $x^2 + y^2 - 2xy = 31^2$, so $2xy = 41^2 - 31^2 = 10 \times 72 = 720$. We obtain $x^2 + y^2 + 2xy = 41^2 + 720 = 2401$, thus $x + y = \sqrt{2401} = 49$. But $x - y = 31$, and so $x = 40$, $y = 9$.

3. How many complex zeros does the polynomial

$$P(z) = z^8 + 2z^6 + 2z^4 - 7z^2 - 8$$

have inside the disk $\{z \in \mathbb{C} \mid |z| < \sqrt[4]{2}\}$?

Solution: Note that $P(z) = Q(z^2)$, where $Q(z) = z^4 + 2z^3 + 2z^2 - 7z - 8$. It is not hard to see that $Q(-1) = 0$, so $P(i) = P(-i) = 0$. We factor $P(z) = (z^2+1)(z^6 + z^4 + z^2 - 8)$. Because of the triangle inequality, for $|z| < \sqrt[4]{2}$,

$$|z^6 + z^4 + z^2 - 8| \geq 8 - |z|^6 - |z|^4 - |z|^2 \geq 8 - 2^{3/2} - 2 - 2^{1/2} = 6 - 3\sqrt{2} > 0.$$

So there are no other zeros inside the disk. Hence the answer to the question is 2.

4. From the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ we randomly choose six numbers. What is the probability that we can write these numbers on the six faces of a cube in such a way that no two adjacent faces contain consecutive numbers? (Two faces are adjacent if they share an edge, one number is written on each face, and the numbers 1 and 9 are considered consecutive.)

Solution: Let us determine which choices allow the required configuration. Write the numbers at the vertices of a nonagon as shown in Figure 1. Then consecutive numbers share a side. Hence the numbers that are at the endpoints of a side, if both chosen, must be written on opposite faces of the cube. So if the number at a vertex is chosen, at most one of the numbers written at its neighboring vertices can be chosen as well.

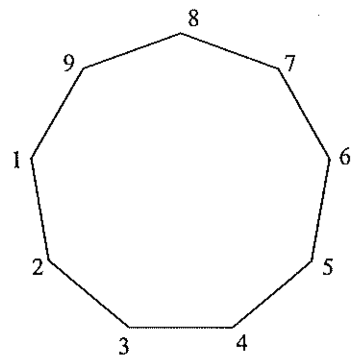


Figure 1

The next observation is that it is not possible that two consecutive numbers cannot be omitted. Indeed, assume that 8 and 9 have been omitted. Then a configuration exists that only contains 1,2,3,4,5,6,7. But then, by the first observation, from each of the triples $(1, 2, 3)$ and $(4, 5, 6)$ a number is missing, and then we only have 5 numbers two choose from.

We conclude therefore that in order for such a configuration to exist, the numbers must be in order, starting somewhere on the nonagon, chosen, chosen, missed, chosen, chosen, missed, chosen, chosen, missed. There are three such configurations: the ones that miss 1,4,7, or 2,5,8, or 3,6,9. For each of them the coloring is possible (see the example for the case where 3,6,9 are missed in Figure 2, where the cube is opened up in the plane).

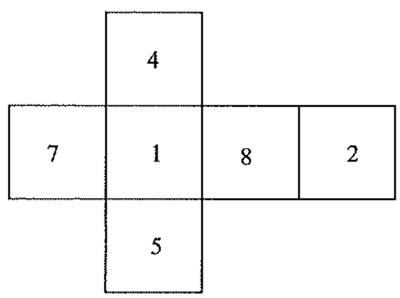


Figure 2

So the probability is

$$\frac{3}{\binom{9}{6}} = \frac{1}{28} \approx .036.$$

5. Let A have coordinates $(2, 3)$ in the plane. Find the coordinates of the points B and C such that the medians from B and C of the triangle ABC have equations $2x + 3y - 1 = 0$ and $2y + 3x - 1 = 0$, respectively.

Solution: The centroid G of the triangle is at the intersection of the two medians, and we find this intersection by solving the system of equations

$$\begin{aligned} 2x + 3y &= 1 \\ 3x + 2y &= 1. \end{aligned}$$

The coordinates are $(x_G, y_G) = (1/5, 1/5)$. Let $B(x_B, y_B)$ and $C(x_C, y_C)$. The coordinates satisfy the system of equations

$$\begin{aligned} &\frac{2 + x_B + x_C}{3} = \frac{1}{5} \\ &\frac{3 + y_B + y_C}{3} = \frac{1}{5} \\ &2x_B + 3y_B = 1 \\ &3x_C + 2y_C = 1. \end{aligned}$$

Here the last two equations express the fact that B and C are on the medians from B and C, respectively. Write the system as

$$\begin{aligned} x_B + x_C &= -\frac{7}{5} \\ y_B + y_C &= -\frac{12}{5} \\ 2x_B + 3y_B &= 1 \\ 3x_C + 2y_C &= 1. \end{aligned}$$

Multiply the first equation by 3, the second by 2 and add to obtain $3x_B + 2y_B + 3x_C + 2x_C = -9$, which combined with the last equation yields $3x_B + 2y_B = -10$. Now combine this with the third equation to obtain $(x_B, y_B) = (-32/5, 23/5)$. From here we find $(x_C, y_C) = (5, -7)$.

7. ~~6.~~ Find, with proof, all positive integers x and y such that

$$\sqrt{x} + \sqrt{y} = \sqrt{2772}.$$

Solution: First note that $2772 = 6^2 \times 77$, so the right-hand side is equal to $6\sqrt{77}$. Rewrite the equation as

$$\sqrt{x} = 6\sqrt{77} - \sqrt{y}$$

and square to obtain $x - 2772 - y = 12\sqrt{77y}$. This implies that $77y$ is a perfect square, so $y = 77m^2$, $m \in \mathbb{N}$. Similarly $x = 77n^2$, $n \in \mathbb{N}$. So we are left to find $(m, n) \in \mathbb{N}^2$ with $m + n = 6$. The solutions are $(1,5)$, $(2,4)$, $(3,3)$, $(4,2)$, $(5,1)$, showing that $(x, y) = (77, ~~385~~ 1925)$, ~~(154, 308)~~ $(308, 1232)$, ~~(231, 231)~~ $(693, 693)$, ~~(308, 154)~~ $(1232, 308)$, $(~~385~~ 1925, 77)$.

6. ~~7.~~ Let

$$f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, \quad f(x) = (x^4 + x^{-4})^4 + (x^4 + x^{-4})^{-4}.$$

Find the intervals on which f is increasing and the intervals on which f is decreasing.

Solution: Write $f(x) = g(g(x))$, where $g(x) = x^4 + x^{-4}$. Using the first derivative test we find that $g(x)$ is increasing on $(-1, 0) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 1)$, and the range of g is $[2, \infty)$. Consequently $f(x)$ is increasing on $(-1, 0) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 1)$.

8. Compute

$$\int_0^2 \sqrt{x^3+1}\,dx + \int_1^3 \sqrt[3]{x^2-1}\,dx.$$

Solution: Consider the function $f : [0,2] \to [1,3]$, $f(x) = \sqrt{x^3+1}$. Its inverse is the function $f^{-1} : [1,3] \to [0,2]$, $f^{-1}(x) = \sqrt[3]{x^2-1}$. Thus we are to compute

$$\int_0^2 f(x)dx + \int_{f(0)}^{f(2)} f^{-1}(x)dx.$$

Using the fact that the graphs of f and f^{-1} are symmetric with respect to the line that bisects the first quadrant, we obtain that the subgraph of f and the subgraph of f^{-1} can be combined to complete the rectangle $[0,2] \times [0,3]$, and so the sum of the two integrals is the area of the rectangle, which is 6.

9. Find the 2020th decimal of the number $(\sqrt{101}+10)^{2019}$.

Solution: Using the binomial expansion we see that the difference

$$(\sqrt{101}+10)^{2019} - (\sqrt{101}-10)^{2019}$$

is an integer. So the 2019th decimal of $(\sqrt{101}+10)^{2019}$ is the same as the 2019th decimal of $(\sqrt{101}-10)^{2019}$. But

$$\sqrt{101}-10)^{2019} = \frac{1}{(\sqrt{101}+10)^{2019}} < \frac{1}{(20)^{2019}} < \frac{1}{10^{2021}},$$

showing that the 2020th decimal of this number is zero.

10. How many triples of integers (a, b, c) are there such that $-2019 < a < b < c < 2019$ and with the property that the lines

$$\begin{aligned} ax + by + c &= 0 \\ bx + cy + a &= 0 \\ cx + ay + b &= 0 \end{aligned}$$

have a common intersection point?

Solution: The lines have a common intersection point if and only if the system of three equations from the statement has a solution. This is equivalent to the fact that the determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

is equal to zero. This happens precisely when $a+b+c = 0$, because the numbers are distinct, so we have to count the number of triples of integers (a, b, c) such that $-2019 < a < b < c < 2019$ and $a+b+c = 0$. Clearly $a < 0$ and $c > 0$, and b can be either positive, in which case $|a| = |b| + |c|$ or negative, in which case $|c| = |a| + |b|$. So the answer is twice the number of triples of non-negative integers (m, n, p), $m, n, p < 2019$ such that $m \neq n$ and $m + n = p$. For fixed p there are $\lceil p/2 \rceil$ solutions. So the answer to the problem is

$$2[1+1+2+2+\cdots 1009+1009] = 2 \times 2 \times \frac{1009 \times 1010}{2} = 2038180.$$