SOLUTIONS IOWA 2017

PROBLEM 1. Solve for x.

The solution set is $\boxed{\{x\colon 2.6 \le x \le 4.6\}}$. We consider three cases: (i) $x < 2.6$, (ii) $2.6 \le x \le 4.6$ and (iii) $x > 4.6$. (i) If $x < 2.6$ then $x - 2.6 < 0$ and $x - 4.6 < 0$ so $|x - 2.6| = -x + 2.6$ and $|x - 4.6| = -x + 4.6$. Thus $|x - 2.6| + |x - 4.6| = -2x + 7.2$, and $-2x + 7.2 = 2 \iff -2x = -5.2 \iff x = 2.6$. Thus there are no solutions with $x < 2.6$.

(ii) Suppose that $2.6 \le x \le 4.6$. Then $|x - 2.6| = x - 2.6$ and $|x - 4.6| = -x + 4.6$, so $|x - 2.6| + |x - 4.6| = x - 2.6 - x + 4.6 = 2$ for all such x.

(iii) Finally, suppose that $x > 4.6$. Then

$$|x - 2.6| + |x - 4.6| = x - 2.6 + x - 4.6 = 2x - 7.2 > 9.2 - 7.2 = 2,$$

so there are no solutions in this case. Thus the solution set is

$$\{x\colon 2.6 \le x \le 4.6\}.$$

PROBLEM 2: Recover blotted out digits.

The only possible values are $\boxed{a = 2,\ b = 4}$, and $6224427 = (99)(62873)$. To find these, consider that the number

$$62ab427 = 6 \cdot 10^6 + 2 \cdot 10^5 + a10^4 + b10^3 + 4 \cdot 10^2 + 2 \cdot 10 + 7$$

must be 0 modulo 9 and 0 modulo 11. Modulo 9 we have

$$0 \equiv 6 + 2 + a + b + 4 + 2 + 7 \equiv a + b + 3,$$

so $a + b \equiv 6 \pmod 9$. Modulo 11 we have

$$6 - 2 + a - b + 4 - 2 + 7 \equiv a - b + 2,$$

so $a - b \equiv 9 \pmod{11}$. Taking into account that a and b are decimal digits we conclude that $a + b = 6$ or 15, and $a - b = 9$ or -2. Also, $a + b$ and $a - b$ are of the same parity (their difference is $2b$), so the possibilities are narrowed to

$$a + b = 15, \qquad a - b = 9$$

or

$$a + b = 6, \qquad a - b = -2.$$

The former case leads to $a = 12,\ b = 3$, so is not a solution. The latter case leads to $a = 2,\ b = 4$, the unique solution.

PROBLEM 3: Two players and 2017 other persons.

Player A has a winning strategy. Initially the number, 2017, of other persons in the circle is odd, so there will be an even number on one of the arcs from A to B and an odd number on the other. This will always be the case when it is A's turn to play, and A wins by always removing a person from the side having an even number of them. If that arc has zero persons, A removes B. If there are other persons on that arc, A removes one, leaving an odd number on both arcs, and B must again leave an even number on one arc and an odd number on the other. The total number of persons is reduced by one on each play, so eventually B leaves zero persons on one of the arcs, and A wins by then removing B.

(If initially the number of other persons is even, then B has a winning strategy, for after A's first play there are an odd number, and B has available the winning strategy described above for A.)

PROBLEM 4: A sum divisible by 11.

Consider the eleven residue classes 0, 1, 2, ..., 10 modulo 11. At least one of these classes contains 11 of the 111 numbers, for if none contained more than 10, there would be no more than 110 numbers in all. With eleven numbers each having the same residue k mod 11, their sum is divisible by 11: (WLOG, a_1, a_2, a_3, $\ldots$, a_{11} each have residue k.)

$$\begin{aligned} a_1 &= 11c_1 + k \\ a_2 &= 11c_2 + k \\ &\vdots \\ a_{11} &= 11c_{11} + k \end{aligned}$$

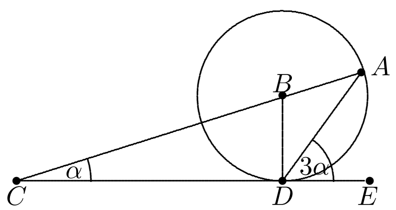
Then $a_1 + a_2 + \cdots + a_{11} = 11(c_1 + c_2 + \cdots + c_{11}) + 11k$, which is a multiple of 11.

PROBLEM 5: The measure of an angle.

The answer is $\boxed{\alpha = 18^\circ}$. Draw the radius BD, which is perpendicular to the tangent line CE. Then angle BDA is $90^\circ - 3\alpha$, and because triangle ABD is isosceles, angle BAD is also $90^\circ - 3\alpha$. Now consider the sum of the angles in triangle ACD:

$$\begin{aligned} 180^\circ &= \angle ACD + \angle CAD + \angle ADC \\ &= \alpha + (90^\circ - 3\alpha) + (180^\circ - 3\alpha). \end{aligned}$$

This simplifies to $5\alpha = 90^\circ$, and we have $\alpha = 18^\circ$.



PROBLEM 6: Factoring 3^{2017}.

There are $\boxed{336}$ such triples. Let $x = 3^p$, $y = 3^q$ and $z = 3^r$. Then $p \le q \le r$, $p+q+r = 2017$, and $3^r < 3^p + 3^q \le 2 \cdot 3^q < 3^{q+1}$, so $r < q + 1$ and therefore $r = q$. Then $p + 2q = 2017$ with $p \le q$, so the possible values of p are 1, 3, 5, …, 671, and $q = r = (2017 - p)/2$. The number of triples is therefore $672/2 = 336$.

PROBLEM 7: Roots in arithmetic progression.

They are $\boxed{m = 6 \text{ and } m = -6/19}$. The sum of the roots is 0 (coefficient of x^3), so to be in arithmetic progression the roots must be of the form $-3a, -a, a, 3a$. (If we write them $a - d, a, a + d, a + 2d$, then their sum is $0 = 4a + 2d$ and $d = -2a$. Then the roots are $a - d = 3a, a, a + d = -a$, and $a + 2d = -3a$.) Thus

$$x^4 - (3m + 2)x^2 + m^2 = (x^2 - 9a^2)(x^2 - a^2) = x^4 - 10a^2x^2 + 9a^4,$$

and we have $m^2 = 9a^4$, $3m+2 = 10a^2$, so $m = \pm 3a^2$. With $m = 3a^2$ we get $10a^2 = 3m+2 = 9a^2 + 2$, and $a^2 = 2$; $m = 6$. In this case the roots are $-3\sqrt{2}, -\sqrt{2}, \sqrt{2}, 3\sqrt{2}$, in arithmetic progression with common difference $2\sqrt{2}$. With $m = -3a^2$ we get $10a^2 = 3m+2 = -9a^2+2$, and $a^2 = 2/19$; $m = -6/19$. In this case the roots are $-3\sqrt{2/19}, -\sqrt{2/19}, \sqrt{2/19}, 3\sqrt{2/19}$, in arithmetic progression with common difference $2\sqrt{2/19}$.

PROBLEM 8: Bound for an integral.

By the mean value theorem we have

$$\int_0^x f(t)dt = \int_0^x (f(t) - f(0))dt = \int_0^x tf'(c_t)dt$$

for some c_t between 0 and t. Because f' is nondecreasing,

$$\int_0^x f(t)dt = \int_0^x tf'(c_t)dt \le \int_0^x tf'(x)dt = \frac{x^2}{2}f'(x)dt.$$

PROBLEM 9: A minimum value.

If, on the contrary, this minimum is greater than 1/4, then each of $r - s^2,\ s - t^2\ t - u^2\ u - r^2$ is greater than 1/4, and hence their sum is greater than 1:

$$1 < (r - s^2) + (s - t^2) + (t - u^2) + (u - r^2) = (r - r^2) + (s - s^2) + (t - t^2) + (u - u^2).$$

This, in turn, implies that

$$\left(r^2 - r + \frac{1}{4}\right) + \left(s^2 - s + \frac{1}{4}\right) + \left(t^2 - t + \frac{1}{4}\right) + \left(u^2 - u + \frac{1}{4}\right) < 0;$$

i.e., that

$$\left(r - \frac{1}{2}\right)^2 + \left(s - \frac{1}{2}\right)^2 + \left(t - \frac{1}{2}\right)^2 + \left(u - \frac{1}{2}\right)^2 < 0.$$

This, however, is impossible, so the minimum in question is less than or equal to 1/4.

PROBLEM 10: Function value at 2017.

We show that $\boxed{f(2017) = 1/2017}$. Note that if $y = f(x)$, they $f(y) = f(f(x)) = 1/f(x) = 1/y$. We are given that $f(4034) = 4033$, and therefore $f(4033) = 1/4033$. Since both 4033 and 1/4033 occur as function values, so does 2017, by the continuity of f and the intermediate value theorem. But if $2017 = f(r)$, then $f(2017) = 1/f(r) = 1/2017$.