22^{nd} Annual Iowa Collegiate Mathematics Competition

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1. Find the greatest k for which $2016 = n_1^3 + n_2^3 + \cdots + n_k^3$, where $n_1, n_2, \ldots, n_k$ are distinct positive integers.

   Solution: Because $1^3 + 2^3 + \cdots + 9^3 = (1 + 2 + \cdots + 9)^2 = 45^2 = 2025$, we see that $2016 = 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3$. We cannot do better than that as $10^3 > 9^3 + 1^3 + 2^3$, so $k = 7$.

2. Evaluate $\dfrac{1}{\log_{16} 2016} + \dfrac{1}{\log_{49} 2016} + \dfrac{1}{\log_{64} 2016} + \dfrac{1}{\log_{81} 2016}$.

   Solution: Because $\log_a 2016 = \frac{\ln 2016}{\ln a}$, our expression is equal to

$$\frac{\ln 16}{\ln 2016} + \frac{\ln 49}{\ln 2016} + \frac{\ln 64}{\ln 2016} + \frac{\ln 81}{\ln 2016} = \frac{\ln(16 \cdot 49 \cdot 64 \cdot 81)}{\ln 2016} = \frac{\ln(2016^2)}{\ln 2016} = 2.$$

3. Consider $14\ldots4$, a "one" followed by n "fours". Find all n for which $14\ldots4$ is a perfect square.

   Solution: We see that $n = 0$ and $n = 2$ are solutions, while $n = 1$ is not. For $n \geq 3$, write $14\ldots4 = 4 \times 361\ldots1$ (with $n - 2$ ones). Then $n = 3$ is a solution and $n > 3$ is not as no perfect square ends in 11 (the next to last digit of a square ending in 1 being even). Indeed, $361\ldots1$, with at least two 1s, is of the form $4k + 3$ and so it cannot be a perfect square. Hence all solutions are $n = 0, 2, 3$.

4. Find all pairs (a, b) of positive real numbers such that $a + b = 1 + \sqrt{1 + \dfrac{a^3 + b^3}{2}}$.

   Solution: We have $(a - b)^2 + (a - 2)^2 + (b - 2)^2 \geq 0$, so $a^2 - ab + b^2 \geq 2(a + b) - 4$. It follows that $\frac{1}{2}(a + b)(a^2 - ab + b^2) \geq (a + b)^2 - 2(a + b)$, implying $1 + \frac{a^3+b^3}{2} \geq (a + b - 1)^2$. The equality holds if and only if $a = b = 2$, so the only solution is $(a, b) = (2, 2)$.

5. Find all primes p such that p^2 divides $5^p - 2^p$.

   Solution: From Fermat's Little Theorem, p divides $5^p - 5$ and p divides $2^p - 2$. Hence p divides $(5^p - 5) - (2^p - 2) = (5^p - 2^p) - 3$. But p divides $5^p - 2^p$, and so p divides 3. It follows that $p = 3$ and indeed 3^2 divides $5^3 - 2^3$.

6. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 2}{n^4 + 4}$.

   Solution: The denominator rewrites as $(n^2 + 2)^2 - (2n)^2 = (n^2 + 2 - 2n)(n^2 + 2 + 2n)$.

   Letting $\dfrac{n^2 - 2}{n^4 + 4} = \dfrac{An + B}{n^2 - 2n + 2} + \dfrac{Cn + D}{n^2 + 2n + 2}$, we obtain $A = \frac{1}{2}$, $B = C = D = -\frac{1}{2}$.

   Hence $\dfrac{n^2 - 2}{n^4 + 4} = \dfrac{1}{2}\left[\dfrac{n - 1}{(n - 1)^2 + 1} - \dfrac{n + 1}{(n + 1)^2 + 1}\right]$, and the sum telescopes to

$$\frac{1}{2}\left[0 + \frac{1}{2} - \lim_{n\to\infty}\left(\frac{n}{n^2 + 1} + \frac{n + 1}{(n + 1)^2 + 1}\right)\right] = \frac{1}{4}.$$

7. Let $A = \begin{pmatrix} 6 & -3 & 2 \\ 15 & -8 & 6 \\ 10 & -6 & 5 \end{pmatrix}$.

(a) Prove that $\det(2I_3 - A) = \frac{1}{\det(A)}$.

(b) Find the least n for which one of the entries of A^n is 2016.

Solution: (a) We have $A^2 = \begin{pmatrix} 11 & -6 & 4 \\ 30 & -17 & 12 \\ 20 & -12 & 9 \end{pmatrix} = 2A - I_3$, implying $(2I_3 - A)\,A = I_3$.

Hence $\det(2I_3 - A)\det(A) = 1$, and the conclusion follows.

(b) The equality $A^2 = 2A - I_3$ implies $(A - I_3)^2 = 0_3$. Hence the matrix $N = A - I_3$ is nilpotent with $N^2 = 0_3$. Then $N^k = 0_3$ for all $k \geq 2$ and so

$$A^n = (N + I_3)^n = nN + I_3 = n \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5n+1 & -3n & 2n \\ 15n & -9n+1 & 6n \\ 10n & -6n & 4n+1 \end{pmatrix}.$$

This can also be shown by examining A^3, making a conjecture, and proving it by induction.

The least n for which one one of the entries of A_n is 2016 is $n = 336$.

8. Solve in real numbers the system of equations $\begin{cases} \sqrt{x}\,(x^2 + 10xy + 5y^2) = 41 \\ \sqrt{2y}\,(5x^2 + 10xy + y^2) = 58. \end{cases}$

Solution: The numbers 5, 10, 10, 5 give us important clues and makes us think of

$$(a \pm b)^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5.$$

Adding the two given equations, after the second one is divided by $\sqrt{2}$, yields $\left(\sqrt{x} + \sqrt{y}\right)^5 = 41 + 29\sqrt{2} = \left(1 + \sqrt{2}\right)^5$, and subtracting, $\left(\sqrt{x} - \sqrt{y}\right)^5 = 41 - 29\sqrt{2} = \left(1 - \sqrt{2}\right)^5$. It follows that $\sqrt{x} + \sqrt{y} = 1 + \sqrt{2}$ and $\sqrt{x} - \sqrt{y} = 1 - \sqrt{2}$, implying $\sqrt{x} = 1$ and $\sqrt{y} = \sqrt{2}$. Hence the (unique) solution is $(x,\, y) = (1,\, 2)$.

9. Evaluate $\displaystyle\int \frac{\left(x^2+1\right)^2}{x^6-1}\,dx$.

Solution: We have $\displaystyle\frac{\left(x^2+1\right)^2}{x^6-1} = \frac{x^4+x^2+1}{x^6-1} + \frac{x^2}{x^6-1} = \frac{1}{x^2-1} + \frac{1}{3}\frac{\left(x^3\right)'}{\left(x^3\right)^2-1}$.

Hence $\displaystyle\int \frac{\left(x^2+1\right)^2}{x^6-1}\,dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{6}\ln\left|\frac{x^3-1}{x^3+1}\right| + C$.

10. Find all continuous functions $f : [0,\, 1] \to \mathbb{R}$ such that $4\int_0^1 f(x)\,dx = \pi + \int_0^1 (1+x^2) f(x)^2\,dx$

Solution: The presence of π and $1 + x^2$ makes us consider the equality $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$.

The given condition rewrites $\displaystyle\int_0^1 \left(\sqrt{1+x^2} f(x) - \frac{2}{\sqrt{1+x^2}}\right)^2 dx = 0$,

implying $\displaystyle\sqrt{1+x^2} f(x) = \frac{2}{\sqrt{1+x^2}}$ for all x in $[0,\, 1]$. Hence $f : [0,\, 1] \to \mathbb{R}$, $f(x) = \frac{2}{1+x^2}$.