22^{nd} Annual Iowa Collegiate Mathematics Competition

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To receive full credit, all problems require complete justification.

1. Find the greatest k for which $2016 = n_1^3 + n_2^3 + \cdots + n_k^3$, where $n_1, n_2, \ldots, n_k$ are distinct positive integers.

2. Evaluate $\dfrac{1}{\log_{16} 2016} + \dfrac{1}{\log_{49} 2016} + \dfrac{1}{\log_{64} 2016} + \dfrac{1}{\log_{81} 2016}$.

3. Consider $14\ldots4$, a "one" followed by n "fours". Find all n for which $14\ldots4$ is a perfect square.

4. Find all pairs (a, b) of positive real numbers such that $a + b = 1 + \sqrt{1 + \dfrac{a^3 + b^3}{2}}$.

5. Find all primes p such that p^2 divides $5^p - 2^p$.

6. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 2}{n^4 + 4}$.

7. Let $A = \begin{pmatrix} 6 & -3 & 2 \\ 15 & -8 & 6 \\ 10 & -6 & 5 \end{pmatrix}$.

   (a) Prove that $\det(2I_3 - A) = \dfrac{1}{\det(A)}$.

   (b) Find the least n for which one of the entries of A^n is 2016.

8. Solve in real numbers the system of equations $\begin{cases} \sqrt{x}\,(x^2 + 10xy + 5y^2) = 41 \\ \sqrt{2y}\,(5x^2 + 10xy + y^2) = 58. \end{cases}$

9. Evaluate $\displaystyle\int \frac{(x^2+1)^2}{x^6 - 1}\,dx$.

10. Find all continuous functions $f : [0, 1] \to \mathbb{R}$ such that

$$4\int_0^1 f(x)\,dx = \pi + \int_0^1 (1 + x^2) f(x)^2\,dx.$$