21st Annual Iowa Collegiate Mathematics Competition
ISU, Saturday, February 21, 2015
Problems by Razvan Gelca

SOLUTIONS

**1.** Asked about his age, a boy replied: "I have a sister, and four years ago when she was born the sum of the ages of my mother, my father, and me was 70 years. Today the sum of the ages of the four of us is twice the sum of the ages that my mother and my father were when I was born." What is the age of the boy?

**Solution:** Let $x$ be the sum of the ages of the parents, and ley $y$ be the age of the boy at the time of the discussion. Then

$$\begin{aligned} x-8+y-4&=70\\ x+y+4&=2(x-2y). \end{aligned}$$

This is equivalent to

$$\begin{aligned} x+y&=82\\ -x+5y&=-4. \end{aligned}$$

Adding the equations we obtain $6y=78$, and we conclude that the age of the boy is $y=13$.

**2.** Let

$$A=\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & 0 \end{pmatrix}.$$

Find, with proof, $A^{2015}$.

**Solution:** It is not hard to guess that for $n\geq 0$,

$$A^{2n+1}=\begin{pmatrix} 0 & 5^n & 0\\ 5^n & 0 & 2\cdot 5^n\\ 0 & 2\cdot 5^n & 0 \end{pmatrix} \quad\text{and}\quad A^{2n+2}=\begin{pmatrix} 5^n & 0 & 2\cdot 5^n\\ 0 & 5^{n+1} & 0\\ 2\cdot 5^n & 0 & 4\cdot 5^n \end{pmatrix}.$$

With the base case $n=0$ easy to check and the induction step

$$\begin{pmatrix} 0 & 5^n & 0\\ 5^n & 0 & 2\cdot 5^n\\ 0 & 2\cdot 5^n & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & 0 \end{pmatrix}=\begin{pmatrix} 5^n & 0 & 2\cdot 5^n\\ 0 & 5^{n+1} & 0\\ 2\cdot 5^n & 0 & 4\cdot 5^n \end{pmatrix}$$

$$\begin{pmatrix} 5^n & 0 & 2\cdot 5^n\\ 0 & 5^{n+1} & 0\\ 2\cdot 5^n & 0 & 4\cdot 5^n \end{pmatrix}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 5^{n+1} & 0\\ 5^{n+1} & 0 & 2\cdot 5^{n+1}\\ 0 & 2\cdot 5^{n+1} & 0 \end{pmatrix}$$

the formulas are proved. Hence

$$A^{2015}=\begin{pmatrix} 0 & 5^{1007} & 0\\ 5^{1007} & 0 & 2\cdot 5^{1007}\\ 0 & 2\cdot 5^{1007} & 0 \end{pmatrix}.$$

Another way to arrive at the final formula is to notice that $A^3=5A$.

**3.** We are given six jugs, the first five containing 2 liters of water each, and the sixth containing one liter. At each step we can select any two jugs, and then pour water from one into another until they contain equal amounts of water. Is it possible to make the quantities of water in all jugs equal? Explain your answer.

**Solution:** If this were possible then at the end each jug would contain one sixth of the whole quantity, that is

$$\frac{2+2+2+2+2+1}{6} = \frac{11}{6} \text{ liters.}$$

Let us start with an experiment. If we combine the last two jugs, then each will contain $\frac{3}{2}$ liters. Now combine one of them with the fourth. The quantities will be

$$2, 2, 2, \frac{7}{4}, \frac{7}{4}, \frac{3}{2}.$$

Combine the third and the fourth to have

$$2, 2, \frac{15}{8}, \frac{15}{8}, \frac{7}{4}, \frac{3}{2}.$$

We notice that the denominators are always powers of 2 and never multiples of 3. Now we can easily prove by induction that this is the case for whatever choice we make. So we can never obtain $\frac{11}{6}$ in one jug, and thus the answer to the problem is negative.

**4.** Find all right triangles whose sides are positive integers and whose perimeter is numerically equal to their area.

**Solution 1:** Let the sides be $a, b, \sqrt{a^2+b^2}$. The condition from the statement translates to

$$a + b + \sqrt{a^2+b^2} = \frac{ab}{2}.$$

After multiplying by 2, separating the square root, and squaring the equality we obtain

$$a^2b^2 + 4a^2 + 4b^2 - 4a^2b - 4ab^2 + 8ab = 4a^2 + 4b^2,$$

which yields

$$ab(ab - 4a - 4b + 8) = 0.$$

Dividing by $ab$ adding an 8 and factoring we obtain

$$(a-4)(b-4) = 8.$$

We have the possibilities $a-4=8, b-4=1$; $a-4=1, b-4=8$; $a-4=2$, $b-4=4$; and $a-4=4$, $b-4=2$. We obtain the triangles $(5,12,13)$ and $(6,8,10)$.

**Solution 2:** We use the characterization of Pythagorean triples as

$$a = k(u^2 - v^2), \quad b = 2kuv, \quad c = k(u^2+v^2)$$

for some $k, u, v$ positive integers, $gcd(u,v) = 1$. The condition from the statement translates into

$$k^2uv(u^2 - v^2) = 2k(u^2 + uv).$$

Dividing by $ku$ and moving $2v$ to the left we obtain

$$v(ku^2 - kv^2 - 2) = 2u.$$

Since $gcd(u,v)=1$, $v=1$ or 2. In the first case we obtain $ku^2-k-2=2u$, that is $ku(u-2)=k-2$. This can only happen for small values of $k$ and $u$, and an easy check yields $k=u=2$. We then obtain the Pythagorean triple $(6,8,10)$.

If $v=2$, then $ku(u-1)=4k+2$, and again this can only happen for small values of $k$ and $u$. An easy check yields $k=1, u=3$, and we obtain the Pythagorean triple $(5,12,13)$.

**5.** Compute

$$\int_0^{\sqrt{\frac{\pi}{3}}} \sin x^2 dx + \int_{-\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{3}}} x^2 \cos x^2 dx.$$

**Solution:** We have

$$\begin{aligned}
&\int_0^{\sqrt{\frac{\pi}{3}}} \sin x^2 dx + \int_{-\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{3}}} x^2 \cos x^2 dx = \int_0^{\sqrt{\frac{\pi}{3}}} \sin x^2 dx + 2\int_0^{\sqrt{\frac{\pi}{3}}} x^2 \cos x^2 dx \\
&= \int_0^{\sqrt{\frac{\pi}{3}}} [\sin x^2 + x(\cos x^2)\cdot 2x] dx = \int_0^{\sqrt{\frac{\pi}{3}}} \frac{d}{dx}(x\sin x^2) dx \\
&= x\sin x^2 |_0^{\sqrt{\frac{\pi}{3}}} = \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

**6.** Let $x_1=\sqrt{3}$ and $x_{n+1}=\sqrt{3}^{x_n}$, $n\ge 1$. Prove that $\lim_{n\to\infty} x_n$ exists and find this limit.

**Solution:** Let

$$f:(0,\infty)\to\mathbb{R}, \quad f(x)=x\ln\sqrt{3}-\ln x.$$

We compute

$$f'(x)=\ln\sqrt{3}-\frac{1}{x}.$$

From here we see that $f'(1/\ln\sqrt{3})=0$, with $f'<0$ before this critical point and $f'<0$ after this critical point. So $1/\ln\sqrt{3}$ is an absolute minimum of $f$. We compute

$$f\left(\frac{1}{\ln\sqrt{3}}\right)=\frac{1}{\ln\sqrt{3}}\ln\sqrt{3}-\ln\frac{1}{\ln\sqrt{3}}=1+\ln(\ln\sqrt{3})=\ln\frac{e\ln 3}{2}>0,$$

because $e\ln 3>e>2$. It follows that $f(x)\ge f(1/\ln\sqrt{3})>0$ for all $x$. In particular $f(x_n)>0$ so

$$x_n\ln\sqrt{3}-\ln x_n=\ln x_{n+1}-\ln x_n>0.$$

It follows that $x_{n+1}>x_n$ for all $n$ so the sequence is increasing and therefore it has a limit $L$. The equality $\sqrt{3}^L=L$ cannot hold for $L$ finite because it would imply, by taking the logarithm, that $f(L)=0$, which we saw above is impossible. So $L=\infty$.

**7.** Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that the equation $f(f(x))=x$ has exactly 5102 solutions and the equation $f(x)=x$ has exactly 2015 solutions?

**Solution:** Assume that such a function exists. Remove the 2015 solutions from $\mathbb{R}$ to obtain a set $A$, and restrict $f$ to $A$. Then $f(f(x))=x$ has $5102-2015=3087$ solutions in $A$ while $f(x)=x$ has none. The 3087 solutions can be grouped in pairs $(x,y)$ such that $f(x)=y$ and $f(y)=x$. But this is impossible since 3087 is an odd number. It follows that such a function does not exist, so the answer to the question is negative.

**8.** Let $P(x) = ax^3 + bx^2 + cx + d$ be a polynomial whose coefficients satisfy

$$a + b + c + 2d > 0 \text{ and } b + d < 0.$$

Show that the equation $P(x) = 0$ has at least one root of absolute value strictly less than 1.

**Solution:** The condition from the statement are equivalent to $P(0)+P(1) > 0$ and $P(1)+P(-1) < 0$. This means that one of the numbers $P(0)$ and $P(1)$ is positive, and one of the numbers $P(1)$ and $P(-1)$ is negative. If $P(1) < 0$ then $P(0) > 0$; the equation $P(x) = 0$ has a root in the interval $(0, 1)$. If $P(1) > 0$ then $P(-1) < 0$, so the equation has a solution in $(-1, 1)$. And if $P(1) = 0$ then $P(0) > 0$ and $P(-1) < 0$, so the equation has a solution in $(-1, 0)$.

**9.** Find all numbers in the interval $[-2015, 2015]$ that can be equal to the determinant of an $11 \times 11$ matrix with entries equal to 1 or $-1$.

**Solution:** Let us consider a matrix with entries equal to $\pm 1$. Its determinant is clearly an integer. Adding the first row to the second, third, ..., eleventh we transform the elements of these rows in either 0 or $\pm 2$. The entries of these new rows are therefore divisible by 2, and factoring these out we deduce that the determinant of the matrix is a multiple of $2^{10}$. There are only three integers that are multiples of $2^{10}$ in the specified interval, namely $0, \pm 2^{10}$. Let us show that each can be the determinant of such a matrix. To obtain 0, just make two rows equal. To obtain $2^{10}$ take the matrix that has 1 on and above the main diagonal, and $-1$ elsewhere. To obtain $-2^{10}$ take the negative of this matrix.

**10.** Find the range of the function

$$f : [-1, 1] \times [-1, 1] \to \mathbb{R}, \quad f(x, y) = x^4 + y^4 + 6x^2y^2 + 8xy.$$

**Solution:** Since the function is continuous, it suffices to find its absolute extrema in the domain $[-1, 1]^2$, and the range will be a closed interval with endpoints the values of these extrema.

First, notice that if we extend the function to the entire plane, then as the distance from a point $(x, y)$ to the origin grows to infinity, so does the value of $f(x, y)$. This means that the (extended) function has an absolute minimum on $\mathbb{R}^2$. We will show that this minimum lies inside $[-1, 1]^2$. We compute

$$\frac{\partial f}{\partial x} = 4x^3 + 12xy^2 + 8y$$
$$\frac{\partial f}{\partial y} = 4y^3 + 12x^2y + 8x.$$

Setting these equal to zero we obtain the system of equations

$$4x^3 + 12xy^2 + 8y = 0$$
$$4y^3 + 12x^2y + 8x = 0.$$

Multiply the first equation by $x$ and the second equation by $y$, then subtract the two equations to obtain $4(x^4 - y^4) = 0$. This can only happen if $x = \pm y$. Returning to the system, one solution is $x = y = 0$, and for any other solution we can only have $x = -y$. Then $4x^3 + 12x^3 - 8x = 0$, so $x = \pm\sqrt{1/2}$. We conclude that the critical points are $(0, 0)$, $(\sqrt{1/2}, -\sqrt{1/2})$ and $(-\sqrt{1/2}, \sqrt{1/2})$. One of these is the point where the function reaches its absolute minimum, and because $f(0, 0) = 0$ and $f(\sqrt{1/2}, -\sqrt{1/2}) = f(-\sqrt{1/2}, \sqrt{1/2}) = -2$, we deduce that the absolute minimum of $f$ on $[-1, 1]^2$ is the same as the absolute minimum of $f$ on $\mathbb{R}^2$, and this is $-2$.

The maximum of $f$ is attained on the boundary, because $f(1, 1) = 16 > 0 = f(0, 0)$. So let us examine the behaviour of $f$ on the boundary. Because of symmetry we only need to analyze the sides $y = 1$ and $y = -1$ of the square. We have $f(x, 1) = x^4 + 6x^2 + 8x + 1$. Its second derivative with respect to $x$ is

$12x^2 + 12$, which is positive, so $f(x, 1)$ is convex. This means that its maximum on $[-1, 1]$ is attained at one of the endpoints of the interval $[-1, 1]$. Repeating the argument we deduce that in order to find the maximum of $f$ we only need to check the four corners: $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$. We deduce that the maximum of $f$ is 16.

We conclude that the range of $f$ is the interval $[-2, 16]$.