21st Annual Iowa Collegiate Mathematics Competition

ISU, Saturday, February 21, 2015

Problems by Razvan Gelca

To receive full credit, all problems require complete justification. Show all your work.

1. Asked about his age, a boy replied: "I have a sister, and four years ago when she was born the sum of the ages of my mother, my father, and me was 70 years. Today the sum of the ages of the four of us is twice the sum of the ages that my mother and my father were when I was born." What is the age of the boy?

2. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

   Find, with proof, A^{2015}.

3. We are given six jugs, the first five containing 2 liters of water each, and the sixth containing one liter. At each step we can select any two jugs, and then pour water from one into another until they contain equal amounts of water. Is it possible to make the quantities of water in all jugs equal? Explain your answer.

4. Find all right triangles whose sides are positive integers and whose perimeter is numerically equal to its area.

5. Compute

$$\int_0^{\sqrt{\frac{\pi}{3}}} \sin(x^2)dx + \int_{-\sqrt{\frac{\pi}{3}}}^{\sqrt{\frac{\pi}{3}}} x^2 \cos(x^2)dx.$$

6. Let $x_1 = \sqrt{3}$ and $x_{n+1} = \sqrt{3}^{x_n}$, $n \geq 1$. Prove that $\lim_{n\to\infty} x_n$ exists and find this limit.

7. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that the equation $f(f(x)) = x$ has exactly 5102 solutions and the equation $f(x) = x$ has exactly 2015 solutions?

8. Let $P(x) = ax^3 + bx^2 + cx + d$ be a polynomial whose coefficients satisfy

$$a + b + c + 2d > 0 \text{ and } b + d < 0.$$

   Show that the equation $P(x) = 0$ has at least one root of absolute value strictly less than 1.

9. Find all numbers in the interval $[-2015, 2015]$ that can be equal to the determinant of an 11×11 matrix with entries equal to 1 or -1.

10. Find the range of the function

$$f : [-1, 1] \times [-1, 1] \to \mathbb{R}, \quad f(x, y) = x^4 + y^4 + 6x^2y^2 + 8xy.$$