Twentieth Annual Iowa Collegiate Mathematics Competition
University of Northern Iowa, Saturday, March 1, 2014

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To receive full credit, all problems require complete justification.

1. Find the point on the parabola $y = x^2 - 4x + 3$ that is closest to the line $y = -2x - 5$.

2. Determine whether there exists an infinite sequence $(a_n)_{n=1}^{\infty}$ of positive numbers, such that the series $\sum_{n=1}^{\infty} s_n$ is convergent, where $s_n = \frac{a_1+a_2+\ldots+a_n}{n}$.

3. Let a and b be positive integers such that $8a = 13b$. Show that the integer $a + b$ is composite.

4. Define the subset of complex numbers $A = \{z : |z| = \sqrt{2}|z - \sqrt{2}|\}$. Find $\max_{z \in A}|z|$.(For a complex number $z = a + bi$, its norm $|z|$ is defined by $|z| = \sqrt{a^2 + b^2}$)

5. Given positive integers $n_1, n_2, \ldots, n_{100}$ such that $\frac{1}{\sqrt{n_1}} + \frac{1}{\sqrt{n_2}} + \cdots + \frac{1}{\sqrt{n_{100}}} = 20$.
Prove that at least two of the integers are equal.

6. Evaluate the integral $\int_{-1}^{1} \frac{dx}{1+x^3+\sqrt{1+x^6}}$.
(Hint: consider the even and the odd part of the given integrand function.)

7. In a certain triangle each height is an integer multiple of the radius of the inscribed circle. Prove that the triangle is equilateral.

8. Determine whether there exist integers a, b, c and d, such that the last four digits of $(a + b)(b + c)(c + d)(d + a)$ are 2014.

9. A bag contains 60 tokens, each one has value of 2, 3, 4, 5 or 6 dollars. Prove that one can choose some of them (without replacement) with the total value of \$60.

10. Let
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$
For a positive integer n, let $F(n)$ denote the sum of the absolute values of all the entries of A^n. Find the smallest n, for which $F(n) \geq 2014$.