NINETEENTH ANNUAL
IOWA COLLEGIATE MATHEMATICS COMPETITION

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Problems

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The problems below are listed in roughly increasing order of difficulty. In particular, the last three are pretty hard. Each problem requires proof or justification. An answer alone is not sufficient. Calculators are permitted, but not particularly useful.

1 Sibling Rivalry

Suppose there are an infinite number of pool balls, which each have a positive integer written on it. For each integer label, there is an infinite supply of balls with that label.

You have a box which contains finitely many such balls. (For example, it may have six #3 balls and twelve #673 balls and a million #2 balls.) Your goal is to empty the box. You may remove any *one* ball you want at each turn. However, whenever you remove a ball, your little brother gets a turn, and he is allowed to add *any finite amount* of balls to the box, as long as each one has a lower number than the one you removed. For example, if you remove one #3 ball, your brother can replace it with 50 #2 balls and 2013 #1 balls.

However, if you remove a #1 ball, then your brother cannot add anything, since there are no balls with lower numbers.

Is it possible to empty the box in a finite number of turns?

2 A Sequence of Squares

Prove that every number in the sequence

$$49, 4489, 444889, 44448889, \ldots$$

is a perfect square.

3 Spin Room

A fan with four equally spaced blades is spinning at 100 revolutions per second clockwise. The fan is in a room that is only illuminated by a strobe light, which flashes (for a tiny instant) 48 times per second. What is the *apparent* speed of the fan, when viewed by a person in the room? In other words, how many revolutions per second does the person actually see?

4 Polynomial Parity

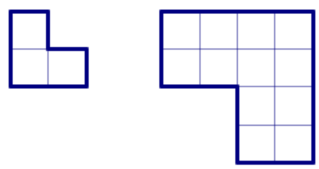
Let $f(x)$ be a polynomial with integer coefficients satisfying $f(2013) = 2012$. Assume that $f(x)$ can be factored into five polynomials with integer coefficients:

$$f(x) = g_1(x)g_2(x)g_3(x)g_4(x)g_5(x).$$

Prove that the sum of the coefficients of at least one factor $g_i(x)$ is odd.

5 A Big Ell

Define a *size-n tromino* to be the shape you get when you remove one quadrant from a $2n \times 2n$ square. In the figure below, a size-1 tromino is on the left and a size-2 tromino is on the right.



We say that a shape can be *tiled with size-1 trominos* if we can cover the entire area of the shape—*and no excess area*—with *non-overlapping* size-1 trominos. For example, it is easy to see that a 2×3 rectangle can be tiled with size-1 trominos, but a 3×3 square cannot be tiled with size-1 trominos.

Can a size-2013 tromino be tiled by size-1 trominos?

6 Playing with a Few Blocks

Is it possible to color the faces of 27 identical $1 \times 1 \times 1$ cubes, using the colors red, white, and blue, so that one can arrange them to form a $3 \times 3 \times 3$ cube with all exterior faces red; and then rearrange them to form a $3 \times 3 \times 3$ cube with all exterior faces blue; and finally, rearrange them to form a $3 \times 3 \times 3$ cube with all exterior faces white?

7 How Abundant are Abundant Numbers?

An *abundant* number is a positive integer n with $\sigma(n) > 2n$, where $\sigma(n)$ denotes the sum of the divisors of n (including 1 and n). For example, 11 is not abundant, because $\sigma(11) = 1 + 11 = 12 < 2 \cdot 11$, but 18 is abundant, because $\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39 > 2 \cdot 18$.

A positive integer less than a googolplex ($10^{10^{100}}$) is chosen randomly, with all choices equally likely. Let P be the probability that the chosen number is abundant.

(a) Prove that $P \geq 1/6$ for partial credit;

(b) Prove that $P > 1/5$ for full credit.

8 The Reciprocal of a Semicircle

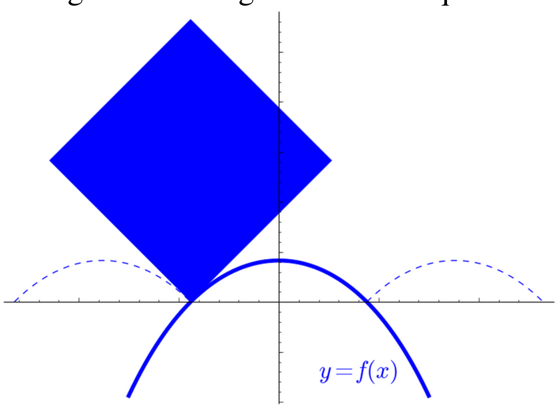
Let S be the semicircle in the complex plane with center at $1/2$ and radius $1/2$, excluding the left endpoint at the origin. In other notation,

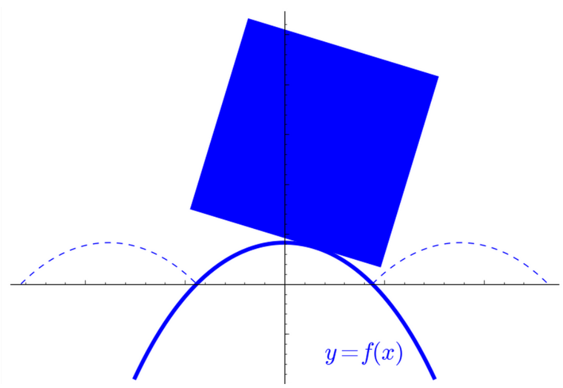
$$S = \left\{ \frac{1}{2} + \frac{1}{2}e^{i\theta} : 0 \leq \theta < \pi \right\}.$$

Describe, with proof, the image of S under the transformation $z \to 1/z$. In other words, describe the set of points $\{1/z : z \in S\}$.

9 Square Wheel

Surprisingly, it's possible to get a smooth ride on a square-wheeled bicycle, provided that the riding surface is the correct shape. In the illustration below, a square with side length 2 rotates clockwise around its center, staying tangent to the indicated riding surface, with no slipping, and the center point of the square always lies on the line $y = \sqrt{2}$! The figure on the left shows the starting point, and the figure on the right shows the square after it has traveled some distance.





The riding surface consists of translated copies of arcs of the part of the graph of $y = f(x)$ that lies above the x-axis. The graph of $y = f(x)$ is the thick-shaded line, and two of the translated arcs are shown with dashed lines.

Find $f(x)$.

10 An Irrational Conclusion

It is well known (and you don't have to prove it) that the set of real numbers forms an infinite-dimensional vector space V over the rationals (i.e., the set of vectors are the real numbers, and the set of scalars are the rational numbers).

Let $2, 3, 5, 7, \ldots, p_{2013}$ be the first 2013 prime numbers, and define the vectors $v_k = \ln(p_k)$ for $k = 1, 2, \ldots, 2013$. Let S be the subspace of V spanned by the vectors $v_1, v_2, \ldots, v_{2013}$. Find, with proof, the dimension of S.