SOLUTIONS IOWA 2012

PROBLEM 1: Logs and exponents.

The solution is $\boxed{\{x : x > 1 \text{ or } x < -5\} = (-\infty, -5) \cup (1, \infty)}$. Using the facts that

$$e^{2\ln|x|} = x^2 \quad \text{and} \quad \log_2 32 = 5,$$

we obtain the equivalent inequality $x^2 + 4x - 5 > 0$; i.e., $(x+5)(x-1) > 0$. This holds iff both factors have the same sign. Both are positive when $x > 1$ and both are negative when $x < -5$, leading to the solution given above.

PROBLEM 2: Image of a function.

The image is $\boxed{(0, 4] = \{x : 0 < x \leq 4\}}$. Rewrite $f(x)$ as follows:

$$\begin{aligned} f(x) &= \frac{(\sqrt{x^2+12x+85} - \sqrt{x^2+12x+45})(\sqrt{x^2+12x+85} + \sqrt{x^2+12x+45})}{\sqrt{x^2+12x+85} + \sqrt{x^2+12x+45}} \\ &= \frac{(x^2+12x+85) - (x^2+12x+45)}{\sqrt{x^2+12x+85} + \sqrt{x^2+12x+45}} \\ &= \frac{40}{\sqrt{(x+6)^2+49} + \sqrt{(x+6)^2+9}}. \end{aligned}$$

The denominator is minimized when $x = -6$, so the maximum value of $f(x)$ is

$$f(-6) = \frac{40}{7+3} = 4.$$

Also, $f(x) > 0$ for all x, and $f(x) \to 0$ as $x \to \infty$. By the continuity of f and the Intermediate Value Theorem we conclude that the image of $\boldsymbol{R}$ under f is the interval $(0, 4]$.

PROBLEM 3: All real zeros of F_{2012} are less than 2.

We prove that for all $n \geq 1$, $F_n(x) \geq F_{n-1}(x) > 0$ for $x \geq 2$. The desired result follows. It is clear that $F_1(x) = x^2 - 3 \geq F_0(x) = 1 > 0$ for all $x \geq 2$. Suppose that $F_k(x) \geq F_{k-1}(x) > 0$ for all $x \geq 2$. Then for $x \geq 2$,

$$\begin{aligned} F_{k+1}(x) &= (x^2-2)F_k(x) - F_{k-1}(x) \\ &\geq 2F_k(x) - F_{k-1}(x) \\ &\geq F_k(x) > 0. \end{aligned}$$

By induction, the claim is established.

PROBLEM 4: Radius of the circle.

We show that $\boxed{r = (2-\sqrt{2})(1-x)}$. The center of the circle is $(1-r, 1-r)$, and the point (x, x) is on the circle, so the distance between these points is r. Thus

$$r^2 = 2(1-r-x)^2.$$

Upon multiplying out on the right and regrouping this may be rewritten

$$r^2 - 4ur + 2u^2 = 0,$$

where $u = 1 - x$. The roots of this quadratic are $r = (2 \pm \sqrt{2})u$. Since $r < u$, the radius is $(2-\sqrt{2})(1-x)$.

PROBLEM 5: A binomial coefficient limit.

The limit is $\boxed{2}$. The sum of the binomial coefficients is 2^n. (For example, it is the total number of subsets of an n-element set.) Then

$$A_n = \frac{2^n}{n+1} \quad \text{and} \quad \sqrt[n]{A_n} = \frac{2}{(n+1)^{\frac{1}{n}}}.$$

It suffices then to show that $\lim_{n\to\infty} a_n = 1$, where $a_n = (n+1)^{1/n}$. We have

$$\ln a_n = \frac{1}{n}\ln(n+1),$$

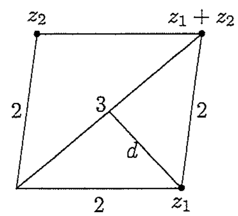
so by L'Hôpital's Rule,

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Then $\lim a_n = 1$, as desired.

PROBLEM 6: Modulus of a complex number.

We show that $\boxed{|z_1 - z_2| = \sqrt{7}}$. In the figure at the right, $|z_1 + z_2|$ is one diagonal of the rhombus, and $|z_1 - z_2|$ is the other diagonal. By the Pythagorean Theorem,



$$d^2 = 2^2 - \left(\frac{3}{2}\right)^2 = \frac{7}{4},$$

so $d = \sqrt{7}/2$, and $|z_1 - z_2| = 2d = \sqrt{7}$.

PROBLEM 7: A 2012 harmonic mean minimum.

The smallest such m is $\boxed{1007}$, and the associated n is $(1007)(1006) = 1\,013\,042$. If (m, n) have harmonic mean 2012 then

$$2012 = \frac{2}{\frac{1}{m} + \frac{1}{n}} = \frac{2mn}{m+n},$$

so $mn - 1006(m+n) = 0$. Then

$$(m - 1006)(n - 1006) = mn - 1006(m+n) + 1006^2 = 1006^2. \qquad (1)$$

We now show that for m and n to be positive and satisfy (1), both must be larger than 1006: First, neither can be equal to 1006, for then the product on the left would be 0. Next, if $0 < m < 1006$, then $0 < 1006 - m < 1006$, and (1) implies that $1006 - n > 1006$, and $n < 0$. Similarly we see that $0 < n < 1006$ is impossible.

Thus, both factors on the left in (1) are positive, and the smallest possible value of m occurs when $m - 1006 = 1$; i.e., $m = 1007$. The corresponding value of n is $1006^2 + 1006 = (1006)(1007)$.

PROBLEM 8: Smallest real part.

We show that in fact $a^2 + b^2$ is real, and the smallest possible value of $a^2 + b^2$ is $\boxed{-1}$. We have $x^2 - rx + r = (x-a)(x-b) = x^2 - (a+b)x + ab$, so $r = a + b = ab$. Then $a^2 + b^2 = (a+b)^2 - 2ab = r^2 - 2r = (r-1)^2 - 1$, which is real because r is real. Also because r is real, the smallest possible value of $(r-1)^2$ is 0, occurring when $r = 1$, and thus the smallest possible value of $a^2 + b^2$ is -1.

PROBLEM 9: The term x_{2012} in a sequence of integers.

We'll show that $\boxed{x_{2012} = 2012^2 a + 2012b}$. Calculation of the first few terms gives $x_1 = a + b$, $x_2 = 4a + 2b$, $x_3 = 9a + 3b$, suggesting that $x_n = n^2 a + nb$ for general n. We prove this by induction. We have it for initial values of n. Suppose that for some k, $x_k = k^2 a + kb$. Then

$$\begin{aligned}
x_{k+1} &= x_k + a + \sqrt{b^2 + 4ax_k} \\
&= k^2 a + kb + a + \sqrt{b^2 + 4a(k^2 a + kb)} \\
&= (k^2 + 1)a + kb + \sqrt{b^2 + 4kab + 4k^2 a^2} \\
&= (k^2 + 1)a + kb + (b + 2ka) \\
&= (k^2 + 2k + 1)a + (k+1)b \\
&= (k+1)^2 a + (k+1)b.
\end{aligned}$$

We conclude by induction that $x_n = n^2 a + nb$ for all n. It is now clear also that every x_n is an integer, and $x_{2012} = 2012^2 a + 2012b$.

PROBLEM 10: Every polynomial is a sum of cubes.

Let A be the set of all polynomials that can be so represented as a sum of cubes of real polynomials. It is clear that A is closed under addition and under multiplication by real constants. But from the fact that $(P_1(x))^3(P_2(x))^3 = (P_1(x)P_2(x))^3$, we see that A is also closed under multiplication (so is an algebra over $\boldsymbol{R}$). Thus if we show that the polynomials 1 and x are in A, it will follow that all polynomials are in A. That 1 is in A is clear. Also, $(x+1)^3 + (-x+1)^3 = 6x^2 + 2$ is in A, and therefore $x^2 \in A$. Then

$$3x = (x+1)^3 - x^3 - 3x^2 - 1$$

is in A, so we have x in A, and the proof is complete.