Eighteenth Annual Iowa Collegiate Mathematics Competition
University of Iowa, Saturday, February 25, 2012

Problems by G. A. Heuer, Concordia College (Moorhead MN)
To receive full credit, all solutions require complete justification.

PROBLEM 1. Logs and exponents

Determine the set of all real numbers x satisfying the inequality

$$e^{2\ln|x|} + 4x - \log_2 32 > 0.$$

Justify your answer by showing all work leading to it.

PROBLEM 2. Image of a function.

Determine the image of the set R of all real numbers under the mapping f defined by

$$f(x) = \sqrt{x^2 + 12x + 85} - \sqrt{x^2 + 12x + 45}.$$

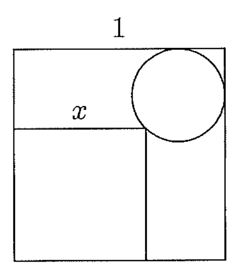
Justify your answer by showing all work leading to it.

PROBLEM 3. All real zeros of F_{2012} are less than 2.

A sequence $\{F_n\}$ of polynomials is defined recursively by $F_0(x) = 1$, $F_1(x) = x^2 - 3$, and for $n \geq 1$, $F_{n+1}(x) = (x^2 - 2)F_n(x) - F_{n-1}(x)$. Prove that if r is real and $F_{2012}(r) = 0$, then $r < 2$.

PROBLEM 4. Radius of the circle.

A square of side $x < 1$ lies inside a unit square as shown at the right. The circle in the upper right corner is tangent to two sides of the larger square and passes through the upper right corner of the smaller square as shown. Express the radius r of the circle as simply as possible as a function of x.



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PROBLEM 5. A binomial coefficient limit.

Let A_n be the arithmetic mean (average) of the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

Determine $\lim_{n\to\infty} \sqrt[n]{A_n}$, and justify your answer. If you know a standard formula for the sum of the binomial coefficients you may use it without proof.

PROBLEM 6. Modulus of a complex number.

Given that z_1 and z_2 are complex numbers with $|z_1| = |z_2| = 2$, and $|z_1 + z_2| = 3$, determine, with proof, $|z_1 - z_2|$.

PROBLEM 7. A 2012 harmonic mean minimum.

The harmonic mean of two positive numbers is the reciprocal of the arithmetic mean of their reciprocals. Determine, with proof, the smallest positive integer m for which there exists a positive integer n such that the harmonic mean of $\{m, n\}$ is 2012.

PROBLEM 8. Smallest real part.

Let r be a real number, and let a, b be the complex solutions of the quadratic equation $x^2 - rx + r = 0$. Determine, with proof, the smallest possible value of the real part of $a^2 + b^2$.

PROBLEM 9. The term x_{2012} in a sequence of integers.

Positive integers a and b are given, and a sequence $\{x_n\}$ is defined recursively by $x_0 = 0$ and for $n \geq 0$, $x_{n+1} = x_n + a + \sqrt{b^2 + 4ax_n}$. Show that every term x_n is an integer, and determine, with proof, the value of x_{2012} (in terms of a and b).

PROBLEM 10. Every polynomial is a sum of cubes.

Prove that for every polynomial $P(x)$ with real coefficients, there exists a positive integer m and polynomials $P_1(x), P_2(x), \ldots, P_m(x)$ with real coefficients such that

$$P(x) = (P_1(x))^3 + (P_2(x))^3 + \cdots + (P_m(x))^3.$$