Answers and Proof Sketches

1. We compute

$$p(p(x)+x) = x^4 + 4x^3 + 11x^2 + 14x + 15 = (x^2 + x + 3)(x^2 + 3x + 5).$$

Now suppose α is a root of $p(x)$. Then α is also a root of $p(p(x)+x)$ since we have $p(p(\alpha)+\alpha) = p(0+\alpha) = 0$.

2. If all the students receive the same score on the first quiz, we're done. Otherwise two students, call them A and B, received different scores on the first quiz, and hence have identical scores on the second quiz. Now consider any other student C, who can't match both of A and B on the first quiz. Hence C must agree with either A or B on the second quiz, and hence also receives this common quiz two score. This logic applies to every student, so we deduce they all have the same quiz two score.

3. Minimizing the area corresponds to maximizing the region of overlap, which is a trapezoid. If the fold occurs at x then the trapezoid area is $3x - \frac{9}{8}x^2$, which has a maximum value of 2 at $x = \frac{4}{3}$. Hence the minimum area is $6 - 2 = 4$.

4. The given property leads to the differential equation $\dfrac{dy}{dx} = -\dfrac{y}{2011}$ with initial condition $y(0) = 2$, hence solution $y = 2e^{-x/2011}$. The desired area is given by

$$\int_0^\infty 2e^{-x/2011}\,dx = 4022.$$

5. The car's velocity in the x-direction is $(10t)\cos(\frac{\pi t}{12})$ fps. (Note use of radians.) Hence its displacement to the east in six seconds is given by

$$\int_0^6 (10t)\cos\left(\frac{\pi t}{12}\right)dt \approx 83.28 \text{ feet.}$$

In the same manner the displacement north is

$$\int_0^6 (10t)\sin\left(\frac{\pi t}{12}\right)dt \approx 145.90 \text{ feet.}$$

Hence it is now 168.0 feet from its starting point.

6. Using partial fractions we may rewrite the integrand as $\dfrac{3}{x+3} + \dfrac{1}{x-1}$. Hence the definite integral evaluates to

$$3\ln\left(\frac{5t+3}{t+3}\right) + \ln\left(\frac{5t-1}{t-1}\right).$$

As $t \to \infty$ this quantity approaches $3\ln 5 + \ln 5 = 4\ln 5$.

7. We will argue that the probability is $\frac{1}{2}$ by strong induction, regardless of the number $n \geq 2$ of seats on the plane. The answer is clearly $\frac{1}{2}$ for $n = 2$. In general, the first person takes his own seat with probability $\frac{1}{n}$, in which case everything works out. The first person takes the last person's seat also with probability $\frac{1}{n}$, which guarantees that she does not get her own seat. Otherwise the first person takes the seat of person k, for some $1 < k < n$. All people before person k sit in their assigned seat, then person k effectively takes the role of the first person, choosing a seat at random. By induction, the final person gets her seat with probability $\frac{1}{2}$ once we reach this stage. Putting everything together gives an overall probability of $\frac{1}{2}$ as well.

8. There are a total of 2011! ways to place 2011 nonattacking rooks on the board without regard to the color of their squares. Now consider the white squares in even and odd-numbered rows/columns separately. The rooks on rows/columns of one parity do not interact with those of the other parity, so we may place them independently. In essence, the problem reduces to placing 1005 rooks on a 1005×1005 board and similarly for 1006 rooks, which may be done in 1005! and 1006! ways. Hence the desired fraction is $1005!1006!/2011! = 1/\binom{2011}{1005}$. (Of course, $1/\binom{2011}{1006}$ is an equally acceptable answer.)

9. Suppose a and b are inverses mod 3^n. Then $ab \equiv 1 \bmod 3^n$ implies $ab \equiv 1 \bmod 3$ as well, meaning that either $a \equiv b \equiv 1 \bmod 3$ or $a \equiv b \equiv 2 \bmod 3$. Hence the factors in the product $(1)(4)(7)\cdots(3^n-2)$ cancel in pairs, leaving just 1. The same logic applies to the product $(2)(5)(8)\cdots(3^n-1)$, leaving just $(3^n-1) \equiv -1 \bmod 3^n$. (Because 1 and $3^n - 1$ are their own inverse; all other inverses come in pairs.)

10. Observe that for $v = (1, 1, \ldots, 1)$ and $v = (-n+1, 1, 1, \ldots, 1)$ we have $Av = 0v$ and $Av = nv$, respectively. Therefore $\lambda = 0$ and $\lambda = n$ are both eigenvalues of A. Given any eigenvalue $\lambda \neq 0, 1, n$ of A with eigenvector v, let v_1 be the first coordinate of v, and define

$$w = (v_2 + \tfrac{v_1}{\lambda-1}, v_3 + \tfrac{v_1}{\lambda-1}, \ldots, v_n + \tfrac{v_1}{\lambda-1}).$$

Then w is defined since $\lambda \neq 1$, and $w \neq 0$ since $\lambda \neq 0, n$. (We omit the details.) One then confirms that $A'w = \lambda w$, so λ is also an eigenvalue for A'.