Fifteenth Annual Iowa Collegiate Mathematics Competition

Iowa State University February 28, 2009

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The problems are listed (roughly) in order of difficulty. Each solution requires a proof or justification. Answers only are not enough. Calculators are allowed but certainly not required.

1. Product 1, Sum 0

The product of 30 integers is one. Can their sum be zero?

Solution: No. Clearly, the only integers in the product are 1 and -1. For the product to be one, there must be an even number of (-1)s. But for the sum to be zero there must be 15 1s and (-1)s.

2. As Easy as 3-4-5

A circle of radius r is inscribed in a right triangle with leg 4r. Prove that the triangle is a 3-4-5 right triangle.

Proof: We may assume that the radius is 1 and that one leg is 4. Let the triangle be ABC with right angle at C and $AC = 4$. Let P be the point of tangency of the incircle with hypotenuse AB, let Q be the point of tangency of the incircle with leg BC, and let R be the point of tangency of the incircle with leg AC. Since the two tangents to a circle from an exterior point have the same length (why?), we have $AP = AR = 3$. Let $BP = BQ = x$.

Then $(3+x)^2 = (1+x)^2 + 4^2$. Solving, we have x=2, and the result follows.

3. Speaking of Pythagoras

Prove that the inradius of any Pythagorean right triangle (a right triangle with integer side lengths) has integer length.

Proof: It suffices to prove the result for primitive Pythagorean right triangles. Let the primitive right triangle be ABC with right angle at C. Then hypotenuse AB has odd length, and one of AC and BC is odd and the other is even. It follows that the perimeter of ABC is even. Let the inradius be R and let P_1 be the point of tangency of the incircle with hypotenuse AB, let P_2 be the point of tangency of the incircle with leg BC, and let P_3 be the point of tangency of the incircle with leg AC. Let $BP_1 = BP_2 = X$, and $AP_1 = AP_3 = Y$. Since $CP_2 = CP_3 = R$, the hypotenuse has length $X+Y$, and the legs have length $X+R$ and $Y+R$. Because the perimeter of ABC is even and equals $2X+2Y+2R$, $X+Y+R$ is an integer, and, since $X+Y$ is an integer, the result follows.

4. Equal Integrals

Let F be a polynomial function of degree $2n$, let G be a polynomial function of degree $2n+1$, and suppose that, for a and for some $d > 0$, $F(a+id) = G(a+id)$ for $i = 0, 1, 2, \ldots, 2n$.

Let $b=a+(2n)d$. Prove that $\int_a^b F(x)\,dx = \int_a^b G(x)\,dx$

Proof: Let $H(x) = G(x) - F(x)$. $H(x)$ has degree $2n+1$ and has zeros at $a+id$ for $i = 0, 1, 2, \ldots, 2n$. Letting $c = a + nd$, we see that H is symmetric about c.

Since $H(c+t) = -H(c-t)$, H is an odd function about c and therefore

$\int_{c-nd}^{c+nd} H(x)\,dx = \int_a^b H(x)\,dx = 0.$ The result follows.

(Note: This result generalizes the curious fact that Simpson's Rule is exact for cubics as well as quadratics.)

5. Φ Fun

Let $\Phi = \dfrac{1+\sqrt{5}}{2}$. Given positive real numbers X and Y with $X > \Phi Y$.

Prove that $\dfrac{X+Y}{X}$ is closer to Φ than $\dfrac{X}{Y}$ is.

Proof: We must show that $|\Phi - X/Y| > |\Phi - (X+Y)/X|$.

Since $X/Y > \Phi$, $Y/X < 1/\Phi = \Phi - 1$, and so $\Phi > 1 + Y/X = (X+Y)/X$, and we want to show that $X/Y - \Phi > \Phi - (X+Y)/X$. This holds iff $2\Phi < X/Y + (X+Y)/X$, $= 1 + X/Y + Y/X$ iff $\sqrt{5} < X/Y + Y/X$ iff $(X/Y)^2 - (X/Y)\sqrt{5} + 1 > 0$ iff $X/Y < (\sqrt{5} - 1)/2$ or $X/Y > (\sqrt{5} + 1)/2 = \Phi$.

Since $X/Y > \Phi$, the result follows.

(Challenge: Prove that the result also holds if $Y < X < \Phi Y$. That is, the result is true for positive real numbers X and Y with X>Y.)

6. *N* up, *N* down

Choose N elements of $\{1, 2, 3, \ldots, 2N\}$ and arrange them in increasing order. Arrange the remaining N elements in decreasing order. Let D_i be the absolute value of the difference of the ith elements in each arrangement. Prove that $D_1 + D_2 + \ldots + D_N = N^2$

Proof: Imagine the numbers 1 through N colored red and the numbers $N+1$ through $2N$ colored blue. The increasing arrangement will consist of some red elements followed by some blue elements. Suppose there are X red elements and so $N-X$ blue elements. The decreasing arrangement will therefore consist of *X* blue elements followed by *N-X* red elements. So the ith elements of the two arrangements will consist of a "red" number paired with a "blue" number. It follows that $D_1 + D_2 + ... + D_N$ will be the sum of the blue numbers minus the sum of the red numbers, that is $D_1 + D_2 + ... + D_N = \big((N+1) + (N+2) + ... + 2N\big) - \big(1 + 2 + ... + N\big) = N^2$

7. Coin Tossing

Al tosses a fair coin *n* times and Babs tosses a fair coin *n+k* times.

Prove that the probability that Al tosses at least as many heads as Babs tosses is

$$\frac{{}_{2n+k}C_0 + {}_{2n+k}C_1 + ... + {}_{2n+k}C_n}{2^{2n+k}}$$

Solution: Call a head tossed by Al or a tail tossed by Babs "G (good for Al)" and a tail tossed by Al or a head tossed by Babs "B (bad for Al)". Imagine a sequence of *2n+k* Gs and Bs, where the first *n* terms represent Al's tosses and the next *n+k* terms represent Babs' tosses. There are 2^{2n+k} such sequences. Al will toss at least as many heads as Babs when the corresponding G-B sequence has no more than *n* Bs. (Why?) The number of such sequences is ${}_{2n+k}C_0 + {}_{2n+k}C_1 + ... + {}_{2n+k}C_n$, so the desired probability is as stated.

8. Function Phenomenon

Let F and G be real valued functions defined on [0,1].

Prove that there exist a and b in [0,1] such that $| ab - F(a) - G(b) | \geq 1/4$.

Proof: Suppose that the result is false. That is, suppose that for all a, b in [0,1],

$| ab - F(a) - G(b) | < 1/4$.

Then for a=b=0 we have

(*) $-1/4 < F(0) + G(0) < 1/4$;

for a=0 and b = 1 we have

(**) $-1/4 < F(0) + G(1) < 1/4$;

for a=1 and b=0 we have

(***) $-1/4 < F(1) + G(0) < 1/4$;

and for a=b=1 we have

(****) $3/4 < F(1) + G(1) < 5/4$.

From (**) and (***) we have $F(0) + F(1) + G(0) + G(1) < 1/2$, while from (*) and (****) we have $1/2 < F(0) + F(1) + G(0) + G(1)$ – a contradiction.

9. On the Fence Post

There are n posts, numbered 1 through n, arranged in a circle, and there are k colors of paint available. Prove that the number of different ways the posts can be painted so that adjacent posts have different colors is $P_n = (k-1)^n + (-1)^n(k-1)$

Proof: Note first that if the posts are arranged in a row (not in a circle), then the number of ways of painting them (with k colors) so that adjacent posts have different colors is $k(k-1)^{n-1}$. Note that P_n is also the number of ways of painting n posts in a row so that adjacent posts have different colors AND the first and last post are also different colors.

Let P_i be the desired number of paintings of i posts, and consider a desired painting of a circular arrangement of (i+1) posts. Suppose that post 1 is painted red. There are two cases to consider; 1) post i is not painted red, and 2) post i is painted red.

For case 1), note that there are P_i ways to paint the first i posts and (k-2) choices for the color of post (i+1). For case 2), note that there are $\left[k(k-1)^{i-1} - P_i\right]$ ways to paint the first i posts and (k-1) choices for the color of post (i+1). Therefore,

$$P_{i+1} = (k-2)P_i + (k-1)\left[k(k-1)^{i-1} - P_i\right] = k(k-1)^i - P_i$$

Since P_2 is easily seen to be $k(k$-1), the result follows by induction.

10. Integer Sticks

A line segment with odd integer length $n = 2k+1$ is randomly cut into three pieces, each of integer length. What is the probability that the three pieces can be formed into a (non-degenerate) triangle?

Solution: Let the cuts be at x and y where $x<y$. Note that there are ${}_{n-1}C_2$ such pairs of cuts. The three pieces, of lengths x, $y-x$, and $n-y$, will form a triangle provided that 1) $x + (y-x) > n-y$, or $y > n/2$; 2) $x + (n-y) > y-x$, or $y-x < n/2$; and 3) $(y-x) + (n-y) > x$, or $x < n/2$.

Consider the lattice grid consisting of the ${}_{n-1}C_2$ points (x,y) with $1 \le x < y \le n-1$. We want to know how many of the points (x,y) satisfy the above three conditions. That is, we want to know how many lattice points are inside (not on the boundary of) the triangular region bounded by

$$y = n/2,\ x = n/2, \text{ and } y = x + n/2.$$

For $n = 2k+1$, there are k^2 points in the lattice grid satisfying $y < n/2$ and $x < n/2$. Of these, $k(k+1)/2$ lie below the line $y = x + n/2$ (and $(k-1)k/2$ lie above). So the desired probability is

$$\frac{\frac{k(k+1)}{2}}{{}_{n-1}C_2} = \frac{n+1}{4(n-2)}.$$

(Challenge: What is the answer if the segment has even integer length?)