Eighth Annual Iowa Collegiate Mathematics Competition

University of Iowa
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PROBLEM 1. The product is a billion.

1. Find a pair of positive integers whose decimal representation contains no zeros and whose product is 1 000 000 000 (1 billion).
2. Show that there is no other such pair.

PROBLEM 2. How many hands?

A group of farmhands need to hoe weeds in two fields. The larger field is twice the size of the smaller one. All of the farmhands work in the larger field for half a day, then are divided into two groups. Half the hands remain in the larger field, and just have time to finish it that day, while the other half move to the smaller field and leave a small area unfinished at the end of the day. How many farmhands are there in all in the original group? (We assume all work at the same rate, which remains constant, and that the number of farmhand-hours needed is proportional to the area to be hoed.)

PROBLEM 3. Magic $3n$-sequences.

A magic $3n$-sequence is a sequence of $3n$ consecutive positive integers such that the sum of the first $2n$ terms is equal to the sum of the remaining n terms. For example, $2, 3, 4, 5, 6, 7, 8, 9, 10$ is a magic 9-sequence, because $2+3+4+5+6+7 = 27 = 8+9+10$.

1. Find a magic 21-sequence.
2. Find all positive integers n for which a magic $3n$-sequence exists, and determine the first term of the sequence as a function of n. Defend your answer.

PROBLEM 4. Find the area.

Find the area of the region S in the (x,y)-plane defined by

$$S = \{(x,y) : (|x|-1)^2 + (|y|-1)^2 \leq 4\}.$$

PROBLEM 5. $(x\text{-}1)^2$ is a factor.

Show that for every integer $n \geq 2$, $(x\text{-}1)^2$ is a factor of $x^n - n(x\text{-}1) - 1$.

PROBLEM 6. A rational tangent.

The angle θ in the (x,y)-plane is formed by line segments OP and OQ, where O is the origin and the coordinates of P and Q in this rectangular coordinate system are rational. Prove that $\tan \theta$ is rational. (Editor's note: During the exam, it was discovered that if $\theta = [(\pi)/ 2]$, then of course $\tan \theta$ is not rational. So this problem must be interpreted only for where $\tan \theta$ is defined.)

PROBLEM 7. A periodic function.

A function f satisfies the equation $f(x+1) + f(x\text{-}1) = \sqrt{2}\, f(x)$, for all real x. Prove that f is periodic.

PROBLEM 8. Maximum number of elements.

Let U be the set of positive integers less than or equal to 2002:

$$U = \{1, 2, 3, ..., 2002\}.$$

Let M be a subset of U with the property that if $x \in M$, then $20x \notin M$. Determine, with proof, the maximum possible number of elements in M.

PROBLEM 9. An integral.

Suppose that f is a continuous function on $[-2, 8]$, such that $f(6\text{-}x) = f(x)$. If

$$\int_{-2}^{8} f(x)\, dx = A,$$

evaluate, with proof,

$$\int_{-2}^{8} x\, f(x)\, dx.$$

PROBLEM 10. Sum the series.

Let x_1 be a positive real number, and for $n \geq 1$, let

$$x_{n+1} = x_n^2 + x_n.$$

Prove that the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{x_k+1}$$

converges, and find its sum.