Seventh Annual Iowa Collegiate Mathematics Competition

Iowa State University
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Problem 1. Side of a triangle.

If the area of an equilateral triangle is ¾, what is the length of each side?

Problem 2. Solve for x.

Find all real solutions of the equation $\lfloor x \rfloor^2 - 5\lfloor x \rfloor - 6 = 0$.
Here $\lfloor x \rfloor$ denotes, as usual, the greatest integer less than or equal to x.

Problem 3. Taylor's Theorem.

According to Taylor's Theorem, if f is a twice differentiable fucntion on an interval containing *a* and *b*, with $a \neq b$, then there is a number *c* between *a* and *b* such that

$$f(b) - f(a) = f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2.$$

Express *c* as simply as possible in terms of *a* and *b* if $f(x) = x^3$.

Problem 4. An irrational number.

Let *r* and *s* be positive rational numbers with $\sqrt{r}$ irrational. Prove that $\sqrt{r} + \sqrt{s}$ is irrational.

Problem 5. Multiplicative inverses.

Let *R* be the ring of integers modulo 2001.For example, in *R*, $1900 + 125 = 24$ and $90^2 = 96$.

(a) Determine whether the element 1334 has a multiplicative inverse in R, and if so, find it. If not show this.

(b) Do the same for the element 1333.

Problem 6. A harmonic identity.

For each positive integer n, let $h(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

Prove that for every integer $n \geq 2$, $n+h(1)+h(2)+\cdots+h(n-1)=nh(n)$.

Problem 7. Sum of squares divisible by *n*.

A certain set of n integers has the property that the difference between the product of any $n-1$ of them and the remaining one is divisible by n.Prove that the sum of the squares of all n integers is divisible by n.

Problem 8. Sum the series.

Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2+3n}{3^n}$, and justify your answer.

Problem 9. A multiple of 49.

Afer serveral applications of the operation of differentiation and the operation of multiplication by $x-1$, performed in unspecified order, the polynomial x^8+x^7 is changed to $ax+b$, where $a \neq 0$.

Prove that $a-b$ is an integer divisible by 49.

Problem 10. Integer roots.

Find all real numbers p such that all three roots of the cubic equation $5x^3-5(p+1)x^2+(71p-1)x+1=66p$ are positive integers.
