First Annual Iowa Collegiate Mathematics Competition

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1. Let a be the integer whose base 10 representation consists of 119 ones:

$$a = \underbrace{111...1}_{119}$$

Prove that a is not prime.

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2. Each cube in a collection of n^3 $1 \times 1 \times 1$ cubes is painted all red or all blue. The cubes are then used to construct an $n \times n \times n$ cube. The resulting cube is completely blue on the outside, but every small cube in the interior is red. The cube is to be reassembled so that the outside surface is completely red. For what positive integer values of n can this be done?

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3. An $n \times n$ matrix A satisfies the equation

$$A^4 + 4A^3 + 6A^2 + 4A + I = Z$$

where I is the $n \times n$ identity matrix and Z is the $n \times n$ zero matrix.

   a. Prove that A is invertible.
   b. Is $A = -I$ the only solution to the equation? Justify your answer.

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4. Find a formula for $\dfrac{d^n}{dx^n} \cos^3 x$.

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5. For x > 0 define $f(x) = \int_1^x \dfrac{\ln t}{1+t} dt.$

Find the value of $f(\sqrt{e}) + f(1/\sqrt{e}).$

6. Let $a_1, a_2, \ldots, a_n$ be real numbers with

$$a_1^2 + a_2^2 + \ldots + a_n^2 = 1.$$

Prove that $-\dfrac{1}{2} \le \displaystyle\sum_{1 \le j < k \le n} a_j a_k \le \dfrac{n-2}{2}.$

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7. Vectors a $\vec{a}$, $\vec{b}$, and $\vec{c}$ are chosen from R^n such that

      (i) For each vector, each of the coordinates is -1 or 1.
      (ii) The vectors are mutually perpendicular.

Prove that this can be done only if *n* is a multiple of 4.

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8. Let $p(x) = x^7 - 3x^6 - 8x^5 - 5x^4 - 7x^3 - 17x^2 - x - 4.$

a. Prove that the equation $p(x) = 0$ has exactly one positive root.

b. Let *r* be the positive root of $p(x) = 0$. Prove that if *s* as any other real or complex root of $p(x) = 0$, then $|s| < r.$

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9. Let *E* be an ellipse in the plane and let *A* be a fixed point inside of *E*. Suppose that two perpendicular lines through *A* intersect *E* in points P, P' and Q, Q' respectively. Prove that

$$\frac{1}{(AP)(AP')} + \frac{1}{(AQ)(AQ')}$$

is independent of the choice of lines.

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10. Let N_0 be the set of positive integers whose decimal expressions ***do not*** contain the digit 0. (Hence $1294 \in N_o$ but $10294 \notin N_o.$). Does the series

$$\sum_{k \in N_o} \frac{1}{k}$$

converge or diverge? Justify your answer.

This page was last revised on October 25, 1997.